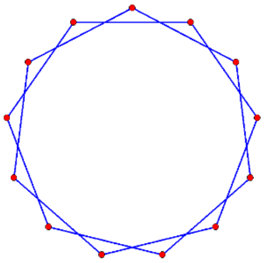
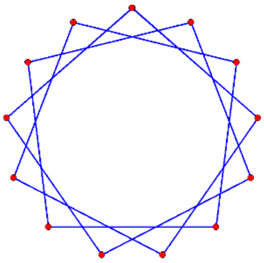
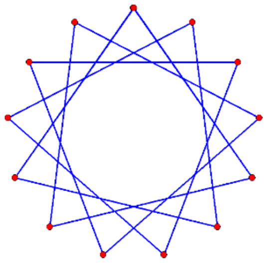
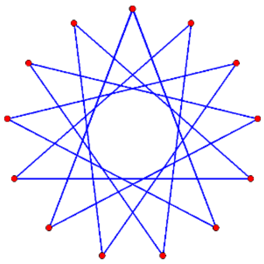
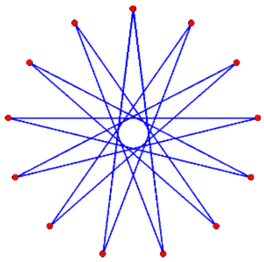
Solutions for “Suggestions for reflection and for investigations”

Chapter 1

✱ A 1.1:

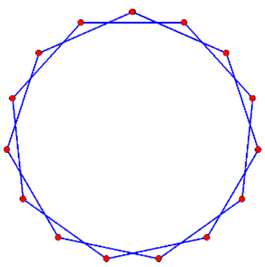
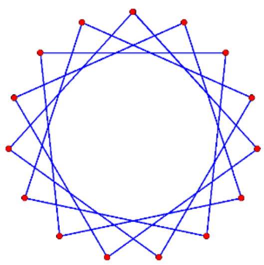
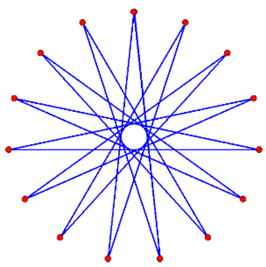
***n* = 13:**

Since *n* is a prime number, all star figures {13/k} with *k* = 2, 3, 4, 5, 6 can be drawn as a closed polygonal lines:

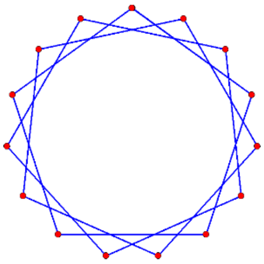
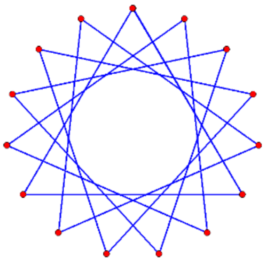
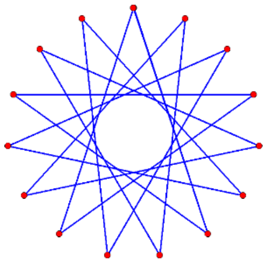
 {13/2} 0 – 2 – 4 – 6 – 8 – 10 – 12 – 1 – 3 – 5 – 7 – 9 – 11 – 0	 {13/3} 0 – 3 – 6 – 9 – 12 – 2 – 5 – 8 – 11 – 1 – 4 – 7 – 10 – 0	 {13/4} 0 – 4 – 8 – 12 – 3 – 7 – 11 – 2 – 6 – 10 – 1 – 5 – 9 – 0
 {13/5} 0 – 5 – 10 – 2 – 7 – 12 – 4 – 9 – 1 – 6 – 11 – 3 – 8 – 0	 {13/6} 0 – 6 – 12 – 5 – 11 – 4 – 10 – 3 – 9 – 2 – 8 – 1 – 7 – 0	

***n* = 15:**

Since *n* = 3 · 5 the star figures {15/*k*} with *k* = 2, 4, 7 (coprime) can be drawn as closed polygonal lines:

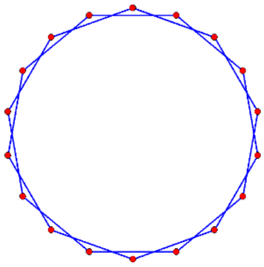
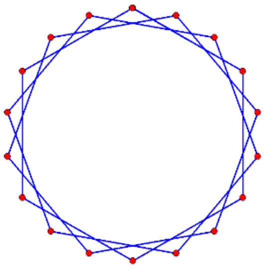
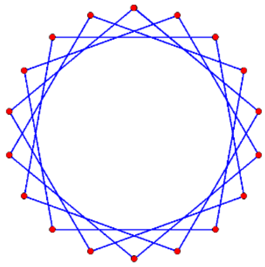
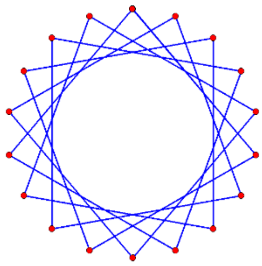
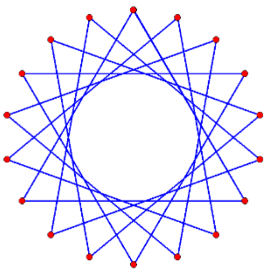
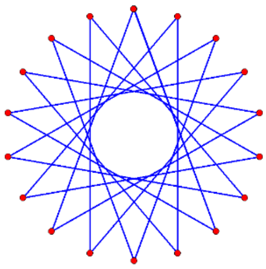
 {15/2} 0 – 2 – 4 – 6 – 8 – 10 – 12 – 14 – 1 – 3 – 5 – 7 – 9 – 11 – 13 – 0	 {15/4} 0 – 4 – 8 – 12 – 1 – 5 – 9 – 13 – 2 – 6 – 10 – 14 – 3 – 7 – 11 – 0	 {15/7} 0 – 7 – 14 – 6 – 13 – 5 – 12 – 4 – 11 – 3 – 10 – 2 – 9 – 1 – 8 – 0

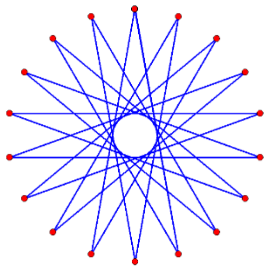
The star figure {15/3} consists of 3 regular 5-sides polygons, since 3 is a divisor of 15, star figure {15/5} consists of 5 regular 3-sided polygons, since 5 is a divisor of 15, and star figure {15/6} consists of three 5-pointed stars of type {5/2} because of gcd(15; 6) = 3:

 {15/3} 0 – 3 – 6 – 9 – 12 – 0, 1 – 4 – 7 – 10 – 13 – 1, 2 – 5 – 8 – 11 – 14 – 2	 {15/5} 0 – 5 – 10 – 0, 1 – 6 – 11 – 1, 2 – 7 – 12 – 2, 3 – 8 – 13 – 3, 4 – 9 – 14 – 4	 {15/6} 0 – 6 – 12 – 3 – 9 – 0, 1 – 7 – 13 – 4 – 10 – 1, 2 – 8 – 14 – 5 – 11 – 2

***n* = 18:**

Since n = 2 · 3 · 3, only the star figures {18/k} with k = 5, 7 (coprime) can be drawn as closed polygonal lines. Star figure {18/2} consists of 2 regular 9-sided polgons, since 2 is a divisor of 18. Star figure {18/3} consists of 3 regular 6-sided polygons, since 3 is a divisor of 18. Star figure {18/4} consists of two 9-pointed stars of type {9/2} because gcd(18;4) = 2. Star figure {18/6} consists of 6 regular triangles, since 6 is a divisor of 18. Star figure {18/8} consists of two 9-pointed stars of type {9/4} because of gcd(18;8} = 2.

 {18/2} 0 – 2 – 4 – 6 – 8 – 10 – 12 – 14 – 16 – 0, 1 – 3 – 5 – 7 – 9 – 11 – 13 – 15 – 17 – 1	 {18/3} 0 – 3 – 6 – 9 – 12 – 15 – 0, 1 – 4 – 7 – 10 – 13 – 16 – 1, 2 – 5 – 8 – 11 – 14 – 17 – 2	 {18/4} 0 – 4 – 8 – 12 – 16 – 2 – 6 – 10 – 14 – 0, 1 – 5 – 9 – 13 – 17 – 3 – 7 – 11 – 15 – 1
 {18/5} 0 – 5 – 10 – 15 – 2 – 7 – 12 – 17 – 4 – 9 – 14 – 1 – 6 – 11 – 16 – 3 – 8 – 13 – 0	 {18/6} 0 – 6 – 12 – 18, 1 – 7 – 13 – 1, 2 – 8 – 14 – 2, 3 – 9 – 15 – 3, 4 – 10 – 16 – 4, 5 – 11 – 17 – 5	 {18/7} 0 – 7 – 14 – 3 – 10 – 17 – 6 – 13 – 2 – 9 – 16 – 5 – 12 – 1 – 8 – 15 – 4 – 11 – 0

 {18/8} 0 – 8 – 16 – 6 – 14 – 4 – 12 – 2 – 10 – 0, 1 – 9 – 17 – 7 – 15 – 5 – 13 – 3 – 11 – 1		

✱ A 1.2:

Star {5/2} can be colored with 2 colors: Inside we have a regular 5-sided polygon = the star {5/1}, outside we have 5 equally sized areas.	Star {7/2} can be colored with 2 colors: Inside we have a regular 7-sided polygon = the star {7/1}, outside we have 7 equally sized areas.	Star {7/3} can be colored with 3 colors: Inside we have the star {7/2} which can be colored with 2 colors, outside we have 7 equally sized areas.
Star {9/2} can be colored with 2 colors: Inside we have a regular 9-sided polygon = the star {9/1}, outside we have 9 equally sized areas.	Star {9/3} can be colored with 3 colors: Inside we have the star {9/2} which can be colored with 2 colors, outside we have 9 equally sized areas.	Star {9/4} can be colored with 4 colors: Inside we have the star {9/3}, which can be colored with 3 colors, outside we have 9 equally sized areas.

Conclusion: Since star {*n*/*k*} consists of star {*n*/*k* - 1} and *n* points on the outside, it can be gradually deduced that the star can be colored with *k* colors.

✱ A 1.3:

A regular 9-sided polygon has ½ · 9 · 6 = 27 diagonals, of which 9 diagonals form the three equilateral triangles of the star {9/3} and 9 diagonals each belong to the regular 9-pointed stars {9/2} and {9/4}.

A regular 10-sided polygon has ½ · 10 · 7 = 35 diagonals, of which 5 diagonals connect only two opposite vertices, 10 diagonals form the two regular 5-sided polygons of the star {10/2}, 10 diagonals form the two 5-pointed stars of which star {10/4} consists and 10 diagonals belong to the regular 10-pointed star {10/3}.

A regular 11-sided polygon has ½ · 11 · 8 = 44 diagonals, of which 11 diagonals each form the regular stars {11/2}, {11/3}, {11/4} and {11/5}.

A regular 12-sided polygon has ½ · 12 · 9 = 54 diagonals, of which 6 diagonals each connect only two opposite vertices, 12 diagonals form the two regular 6-sided polygons of the star {12/2}, 12 diagonals form the three squares of the star {12/3}, 12 diagonals form the two regular 6-sided polygons of which star {12/4} consists and 12 diagonals belong to the regular 12-pointed star {12/5}.

Generalization:

The number of diagonals in a regular *n*-sided polygon is ½ · *n* · (*n* – 3).

For odd *n*, the factor *n* – 3 is an even number and can be divided by 2. ½ · (*n* – 3) thus indicates how many stars can be formed: With *n* = 5 this is ½ · (*n* – 3) = 1 star, with *n* = 7 these are ½ · (*n* – 3) = 2 stars, with *n* = 9 these are ½ · (*n* – 3) = 3 stars etc.

For even *n*, ½ · *n* diagonals are useless for drawing stars, because they only connect two opposite vertices. So ½ · *n* · (*n* – 3) – ½ · *n* = ½ · *n* · (*n* – 4) diagonals remain. ½ · (*n* – 4) indicates how many stars can be formed: With *n* = 6 this is ½ · (*n* – 4) = 1 star, with *n* = 8 these are ½ · (*n* – 4) = 2 stars, with *n* = 10 these are ½ · (*n* - 4) = 3 stars, and so on.

Which star types are created depends – as explained – on *k*: a star made of *k* polygons with *n*/*k* vertices or a star made of *g* stars with *n*/*g* vertices or a star that consists of one closed polygonal lines.

✱ A 1.4:

The “tip” lies "above" a diagonal of the 8-sided polygon, which connects one vertex with the next but one vertex. Therefore, the angle at the “tip” is half as large as the corresponding central angle, namely half as large as $2 \cdot 360°/8$, i.e. $\varepsilon = 45°$.	The “tip” lies "over" one side of the 9-sided polygon. Therefore the angle at the “tip” is half as large as the corresponding central angle, namely half as large as 360°/9, i.e. $\varepsilon = 20°$.	The “tip” lies "above" a diagonal of the 10-sided polygon, which connects one vertex with the fourth next vertex. Therefore, the angle at the “tip” is half as large as the corresponding central angle, namely half as large as $4 \cdot 360°/10$, i.e. $\varepsilon = 72°$.	The “tip” lies "above" a diagonal of the 12-sided polygon, which connects one vertex with the fifth next vertex. Therefore, the angle at the “tip” is half as large as the corresponding central angle, namely half as large as $5 \cdot 360°/10$, i.e. $\varepsilon = 75°$.

✱ A 1.5:

There are peripheral angles "above" and "below" the chord. These complement each other to 180°. The central angle, which belongs to a peripheral angle "below" the chord, is the complementary angle to the central angle, which belongs to a peripheral angle "above" the chord – these two central angles thus complement each other to 360°.

The “tip” is above the diagonal of the 9-sided polygon, which connects one vertex with the fifth next vertex. Therefore, the peripheral angle ε is half as large as the corresponding central angle, namely half as large as $5 \cdot 360°/9$, i.e. $\varepsilon = 100°$.

✱ A 1.6:

The central angle is always greater than 180°.

In the figure on the left, the “tip” is “above” the diagonal of the 11-sided polygon, which connects one vertex with the seventh next vertex. Therefore, the peripheral angle is half as large as the corresponding central angle, namely half as large as $7 \cdot 360°/11$, i.e. $\varepsilon \approx 114.5°$.

In the illustration on the right, the “tip” is “above” the diagonal of the 12-sided polygon, which connects one vertex with the eight-next vertex. Therefore, the peripheral angle is half as large as the corresponding central angle, namely half as large as $8 \cdot 360°/12$, i.e. $\varepsilon = 120°$.

✱ A 1.7:

Since every star of type $\{n/2\}$ contains a regular n-sided polygon inside, i.e. a star of type $\{n/1\}$, the triangles are "put on".

✱ A 1.8:

$x_1 = 1$ and $x_2 = -1$ are solutions of the equation $x^6 - 1 = 0$. Therefore the division $(x^6 - 1) : (x^2 - 1)$ is possible and one obtains $(x^4 + x^2 + 1)$ as the supplement factor.

By multiplying you can check that $(x^4 + x^2 + 1)$ can be represented as product $(x^2 + x + 1) \cdot (x^2 - x + 1)$.

You can use the binomial formula:

$(x^2 + x + 1) \cdot (x^2 - x + 1) = ((x^2 + 1) + x) \cdot ((x^2 + 1) - x) = (x^2 + 1)^2 - x^2 = x^4 + 2x^2 + 1 - x^2$

The solutions of the quadratic equations $x^2 + x + 1 = 0$ and $x^2 - x + 1 = 0$ are:

$$x_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i \; ; \; x_4 = -\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i \; ; \; x_5 = \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i \; ; \; x_6 = \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i$$

So you get the following coordinates

$$(1, 0) ; (-1, 0) ; (-\frac{1}{2}, \frac{\sqrt{3}}{2}) ; (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) ; (\frac{1}{2}, \frac{\sqrt{3}}{2}) ; (\frac{1}{2}, -\frac{\sqrt{3}}{2})$$

✱ A 1.9:

$x_1 = 1$ and $x_2 = -1$ are solutions of the equation $x^8 - 1 = 0$. Therefore the division $(x^8 - 1) : (x^2 - 1)$ is possible and one obtains $(x^6 + x^4 + x^2 + 1)$ as the supplement factor.

After it has been established that $x_3 = i$ und $x_4 = -i$ are solutions of the equation $x^8 - 1 = 0$, too, also the division by $(x^2 + 1)$ must be possible, i. e.

$(x^8 - 1) : (x^2 - 1) \cdot (x^2 + 1) = (x^8 - 1) : (x^4 - 1) = (x^4 + 1)$

By multiplication of $(x^2 + \sqrt{2}\,x + 1) \cdot (x^2 - \sqrt{2}\,x + 1)$ one can proof, that this results in $(x^4 + 1)$.

So we have the following solutions of the quadratic equations $x^2 + \sqrt{2}\,x + 1 = 0$ and $x^2 - \sqrt{2}\,x + 1 = 0$:

$$x_5 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot i \;;\; x_6 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot i \;;\; x_7 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot i \;;\; x_8 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot i$$

So we get the following coordinates

$$(1, 0) ; (-1, 0) ; (0, 1) ; (0, -1) ; \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) ; \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) ; \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) ; \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

✱ A 1.10:

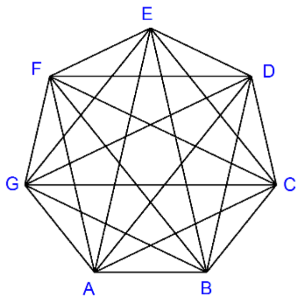
4th round: A – E, D – F, C – G, B – H

5th round: A – F, E – G, D – H, B - C

6th round: A – G, F – H, B – E, C – D

7th round: A – H, B – G, C – F, D – E

✱ A 1.11:.



As in the tournament with 8 teams, you draw a regular 7-sided polygon. The games of a matchday result from an individual side of the polygon and the diagonals parallel to it; the vertex opposite to the individual side marks the pausing team.

1st round: A – B, C – G, D – F, E is free from play

2nd round: B – C, A – D, E – G, F is free from play

3rd round: C – D, B – E, A – F, G is free from play

4th round: D – E, C – F, B – G, A is free from play

5th round: E – F, D – G, A – C, B is free from play

6th round: F – G, A – E, B – D, C is free from play

7th round: G – A, B – F, C – E, D is free from play

✱ A 1.12:

From the following symmetrical table you can see that 15 games are taking place. All possible combinations are noted in the first row and the first column (in alphabetical order).

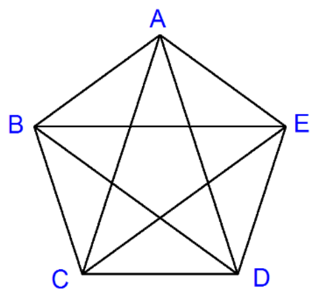
Inside the table there is an "X" if a match pairing is not possible because a player appears twice. The table is symmetrical, because with each pairing the reverse pairing is possible, too.

	AB	AC	AD	AE	BC	BD	BE	CD	CE	DE
AB	X	X	X	X	X	X	X			
AC	X	X	X	X	X			X	X	
AD	X	X	X	X		X		X		X
AE	X	X	X	X			X		X	X
BC	X	X			X	X	X	X	X	
BD	X		X		X	X	X	X		X
BE	X			X	X	X	X		X	X
CD		X	X		X	X		X	X	X
CE		X		X	X		X	X	X	X
DE			X	X		X	X	X	X	X

To get a nice schedule, you should proceed geometrically.

If you look at a regular pentagon, you see that the sides and the diagonals correspond to a two man-team.

If you choose the player which is free from play, then you can determine the teams playing together according to a geometrical pattern:



Opposite to the vertex (free from play) there is a trapezoid, from this choose the two parallel sides:

free from play	Team No. 1	Team No. 2
A	CD	BE
B	DE	AC
C	AE	BD
D	AB	CE
E	BC	AD

Opposite to the vertex (free from play) there is a trapezoid, from this select the non-parallel sides:

free from play	Team No. 1	Team No. 2
A	BC	DE
B	CD	AE
C	DE	AB
D	AE	BC
E	AB	CD

Opposite to the vertex (free from play) there is a trapezoid, from this choose the two diagonals:

free from play	Team No. 1	Team No. 2
A	BD	CE
B	CE	AD
C	AD	BE
D	BE	AC
E	BD	AC

Chapter 2

✱ A 2.1:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\sum$	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190	210

✱ A 2.2:

To determine these limits, a quadratic (in)equation must be solved:

$\frac{1}{2} \cdot n \cdot (n+1) \geq 100 \Leftrightarrow n^2 + n \geq 200 \Leftrightarrow (n + \frac{1}{2})^2 \geq 200.25 \Leftrightarrow n \geq 13.65 \Leftrightarrow n \geq 14$

$\frac{1}{2} \cdot n \cdot (n+1) \geq 1.000 \Leftrightarrow n^2 + n \geq 2.000 \Leftrightarrow (n + \frac{1}{2})^2 \geq 2000.25 \Leftrightarrow n \geq 44.22 \Leftrightarrow n \geq 45$

$\frac{1}{2} \cdot n \cdot (n+1) \geq 1.000.000 \Leftrightarrow n^2 + n \geq 2.000.000 \Leftrightarrow (n + \frac{1}{2})^2 \geq 2000000.25 \Leftrightarrow n \geq 1413.71 \Leftrightarrow n \geq 1414$

✱ A 2.3:

The squares shown in the figure contain eight triangles, which represent the triangular numbers; in addition, there is a dot in the center. So the following relationships are shown in the figures:

$8 \cdot \Delta_3 + 1 = 7^2$; $8 \cdot \Delta_4 + 1 = 9^2$; $8 \cdot \Delta_5 + 1 = 11^2$

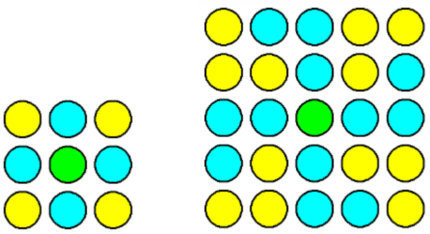
General relationship:

$8 \cdot \Delta_n + 1 = (2n + 1)^2$

To proof this you can use $\Delta_n = \frac{1}{2} \cdot n \cdot (n+1) = \frac{1}{2} \cdot n^2 + \frac{1}{2} \cdot n$:

$8 \cdot \Delta_n + 1 = 4 \cdot n^2 + 4 \cdot n + 1 = (2n + 1)^2$

Figures for n = 1 and n = 2:



✱ A 2.4:

- In the figures you see:

on the left: $3^2 - 3 = 6$ red, $6^2 - 6 = 30$ yellow, $2 \cdot (3 \cdot 6) = 36$ orange colored stones;

in the middle: $6^2 - 6 = 30$ red, $10^2 - 10 = 90$ yellow, $2 \cdot (6 \cdot 10) = 120$ orange colored stones;

on the right: $10^2 - 10 = 90$ red, $15^2 - 15 = 210$ yellow, $2 \cdot (10 \cdot 15) = 300$ orange colored stones

Generally: $(r^2 - r) + (g^2 - g) = 2 \cdot r \cdot g \Leftrightarrow g^2 - 2rg + r^2 = r + g \Leftrightarrow (g - r)^2 = r + g$

If g and r (with g > r) are two consecutive triangle numbers Δ_n and Δ_{n-1}, then their difference is equal to n. According to formula (2.2) we have: $\Delta_{n-1} + \Delta_n = n^2$. So we get

The probability of the event *The two balls have the same color* can be calculated as follows:

$$P((r, r), (g, g)) = \frac{r \cdot (r-1) + g \cdot (g-1)}{(r+g) \cdot (r+g-1)} = \frac{r^2 + g^2 - r - g}{(r+g) \cdot (r+g-1)}$$

The probability of the event *The two balls have different colors* can be calculated as follows:

P((r|g), (g|r)) = $\dfrac{2 \cdot r \cdot g}{(r+g)\cdot(r+g-1)}$

A fair game is given, if both events have the same probabilty, i. e.:

$$\frac{r^2+g^2-r-g}{(r+g)\cdot(r+g-1)} = \frac{2\cdot r\cdot g}{(r+g)\cdot(r+g-1)},$$

$r^2 - 2rg + g^2 - r - g = 0$,

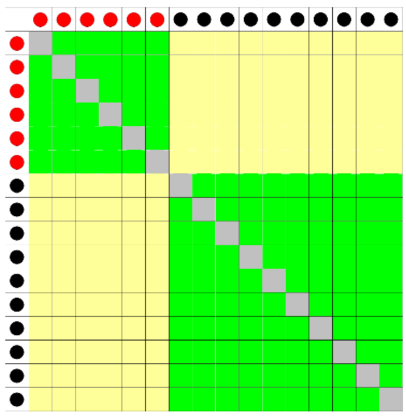
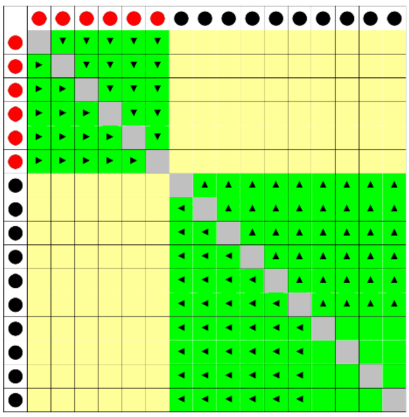
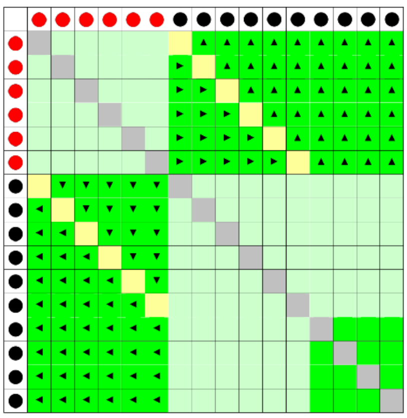
$(r - g)^2 = r + g$ or $(g - r)^2 = r + g$

Note: The following table shows the probability of winning if you bet on the event. You can see that it would be disadvantageous to bet on the event if the numbers of red and yellow balls differ only slightly.

	1	2	3	4	5	6	7	8	9	10
1	0,000	0,333	0,500	0,600	0,667	0,714	0,750	0,778	0,800	0,818
2	0,333	0,333	0,400	0,467	0,524	0,571	0,611	0,644	0,673	0,697
3	0,500	0,400	0,400	0,429	0,464	0,500	0,533	0,564	0,591	0,615
4	0,600	0,467	0,429	0,429	0,444	0,467	0,491	0,515	0,538	0,560
5	0,667	0,524	0,464	0,444	0,444	0,455	0,470	0,487	0,505	0,524
6	0,714	0,571	0,500	0,467	0,455	0,455	0,462	0,473	0,486	0,500
7	0,750	0,611	0,533	0,491	0,470	0,462	0,462	0,467	0,475	0,485
8	0,778	0,644	0,564	0,515	0,487	0,473	0,467	0,467	0,471	0,477
9	0,800	0,673	0,591	0,538	0,505	0,486	0,475	0,471	0,471	0,474
10	0,818	0,697	0,615	0,560	0,524	0,500	0,485	0,477	0,474	0,474
11	0,833	0,718	0,637	0,581	0,542	0,515	0,497	0,485	0,479	0,476
12	0,846	0,736	0,657	0,600	0,559	0,529	0,509	0,495	0,486	0,481
13	0,857	0,752	0,675	0,618	0,575	0,544	0,521	0,505	0,494	0,486
14	0,867	0,767	0,691	0,634	0,591	0,558	0,533	0,515	0,502	0,493
15	0,875	0,779	0,706	0,649	0,605	0,571	0,545	0,526	0,511	0,500

The drawing of two balls from an urn can be illustrated with the help of a combination table:

As the first drawn ball is not put back, a ball cannot be drawn twice (grey fields). The combinations belonging to the event *The balls have the same color* are illustrated by the green fields. The arrowheads indicate that these cells can be moved into yellow cells. The green cells that have not been moved obviously fit into the "free" cells at the top right and bottom left:

✱ A 2.5:

Four triangles with *n* rows are shown, by which the sum 1 + 3 + 5 + ... + (2n + 1) is represented. Obviously the following applies:

$4 \cdot [1 + 3 + 5 + \ldots + (2n-1)] = [(2n-1) + 1]^2 = (2n)^2 = 4n^2$, also $1 + 3 + 5 + \ldots + (2n+1) = n^2$

✱ A 2.6:

In the figure you can see two sums of consecutive odd numbers

(1 + 3) + 1, (1 + 3 + 5) + (3 + 1), (1 + 3 + 5 + 7) +(5 + 3 + 1), (1 + 3 + 5 + 7 + 9) +(7 + 5 + 3 + 1),

so for n = 1, 2, 3, 4 we have

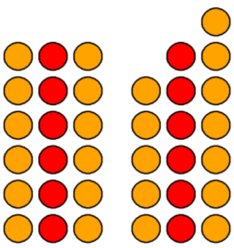
$[1 + 3 + \ldots + (2n - 1) + (2n + 1)] + [(2n - 1) + \ldots + 3 + 1] = (n + 1)^2 + n^2$ according to formula (2.3)

✱ A 2.7:

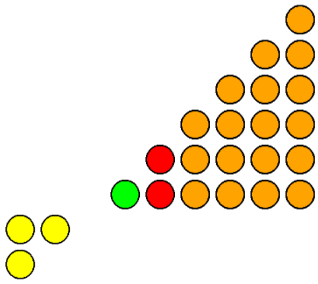
The triangle consisting of 2n lines is composed of four smaller triangles: three triangles with n lines (red, green, blue) and one triangle with n – 1 lines (yellow). If you add a row with n dots to this smaller triangle, you get a triangle with n rows.

✱ A 2.8:

- The number 18 has two odd divisors: 3 and 9. This can be illustrated by the following two figures:



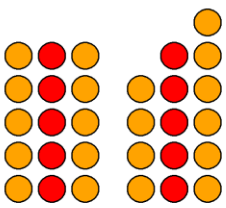
5 + 6 + 7 = 18

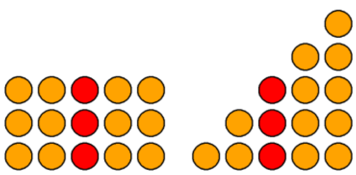


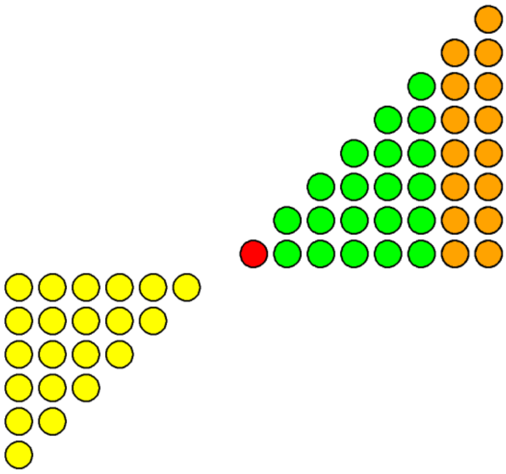
3 + 4 + 5 + 6 = 18

- The number 15 has three odd divisors: 3, 5 and 15. This can be illustrated by the following three figures:

4 + 5 + 6 = 15, 1 + 2 + 3 + 4 + 5 = 15, 7 + 8 = 15

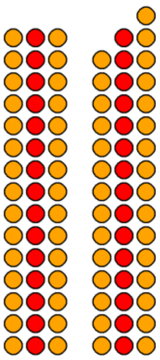


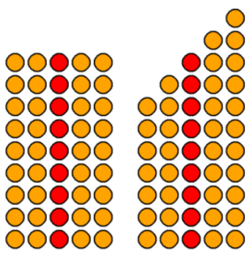


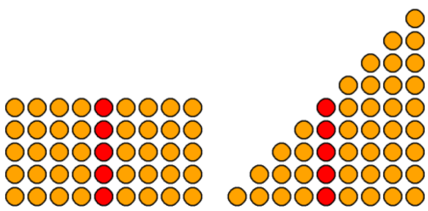


- The number 45 has 5 odd divisors: 3, 5, 9, 15 and 45. This can be illustrated by the following five figures:

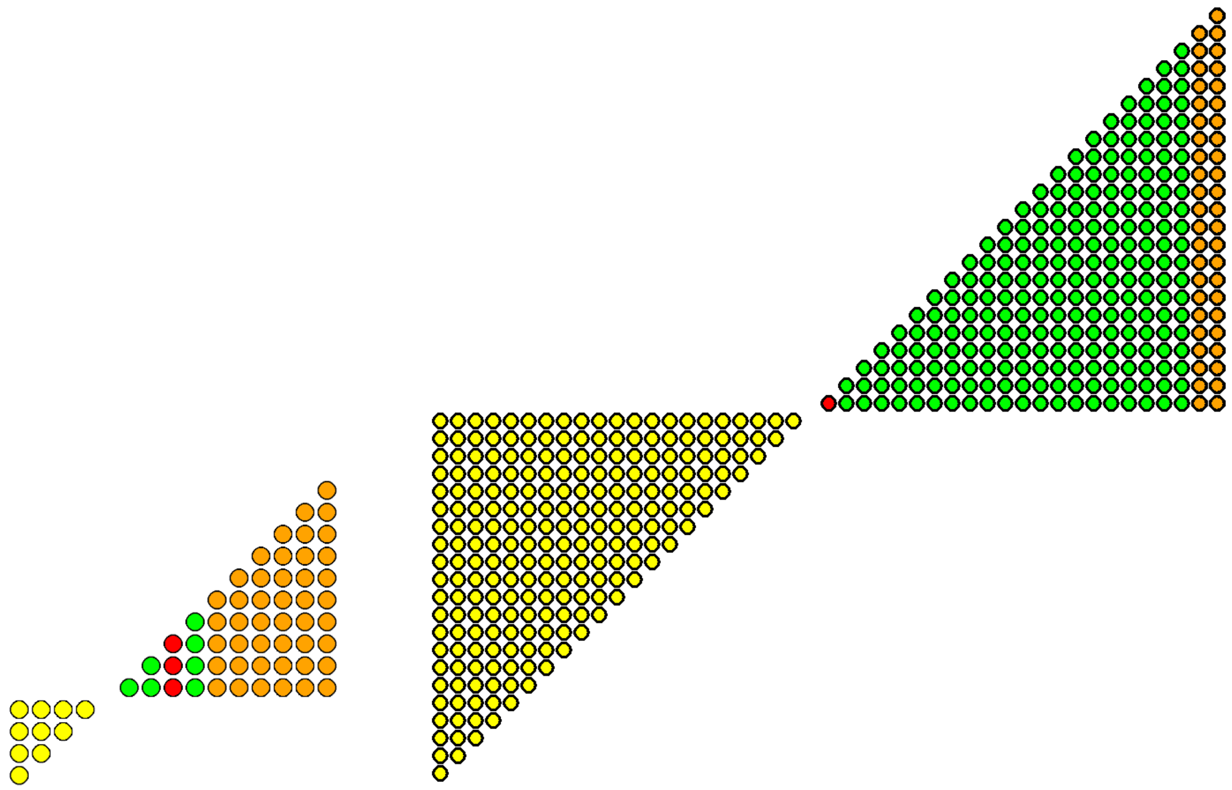
14 + 15 + 16 = 45;   7 + 8 + 9 + 10 + 11 = 45;   1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 =45;







5 + 6 + 7 + 8 + 9 + 10 = 45;   22 + 23 = 45



✱ A 2.9:

next page

✱ A 2.10:

- 2016 has 5 odd divisors: 3, 7, 9, 21, 63. Therefore we get the following 5 representations:

  2016 = 671 + 672 + 673

  = 285 + 286 + 287 + 288 + 289 + 290 + 291

  = 220 + 221 + 222 + 223 + 224 + 225 + 226 + 227 + 228

  = 86 + 87 + … + 95 + 96 + 97 + … + 106

  = 1 + 2 + … + 32 + 33 + … + 64
- 2017 is a prime number, i. e. only one divisor,
- $2018 = 2 \cdot 1009$ has only one odd divisor (1009 is a prime numer),
- $2019 = 3 \cdot 673$ has three odd divisors,
- $2020 = 2^2 \cdot 5 \cdot 101$ has three odd divisors,
- $2021 = 43 \cdot 47$ has three odd divisors,
- $2022 = 2 \cdot 3 \cdot 337$ has three odd divisors,
- $2023 = 7 \cdot 17^2$ has five odd divisors,
- $2024 = 2^3 \cdot 11 \cdot 23$ has three odd divisors.
- $2025 = 3^4 \cdot 5^2$ has 14 odd divisors.

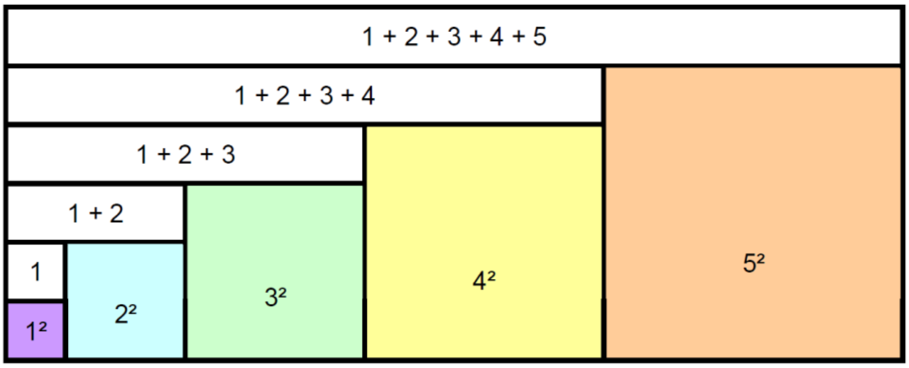
✱ A 2.9:

- Overview over the number of odd divisors for n = 3, 4, …, 100

n	odd divisors (> 1)					number of odd divisors
3	3					1
4						0
5	5					1
6	3					1
7	7					1
8						0
9	3	9				2
10	5					1
11	11					1
12	3					1
13	13					1
14	7					1
15	3	5	15			3
16						0
17	17					1
18	3	9				2
19	19					1
20	5					1
21	3	7	21			3
22	11					1
23	23					1
24	3					1
25	5	25				2
26	13					1
27	3	9	27			3
28	7					1
29	29					1
30	3	5	15			3
31	31					1
32						0
33	3	11	33			3
34	17					1
35	5	7	35			3
36	3	9				2
37	37					1
38	19					1
39	3	13	39			3
40	5					1
41	41					1
42	3	7	21			3
43	43					1
44	11					1
45	3	5	9	15	45	5
46	23					1
47	47					1
48	3					1
49	7	49				2
50	5	25				2
51	3	17	51			3

n	odd divisors (> 1)					number of odd divisors
52	13					1
53	53					1
54	3	9	27			3
55	5	11	55			3
56	7					1
57	3	19	57			3
58	29					1
59	59					1
60	3	5	15			3
61	61					1
62	31					1
63	3	7	9	21	63	5
64						0
65	5	13	65			3
66	3	11	33			3
67	67					1
68	17					1
69	3	23	69			3
70	5	7	35			3
71	71					1
72	3	9				2
73	73					1
74	37					1
75	3	5	15	25	75	5
76	19					1
77	7	11	77			3
78	3	13	39			3
79	79					1
80	5					1
81	3	9	27	81		4
82	41					1
83	83					1
84	3	7	21			3
85	5	17	85			3
86	43					1
87	3	29	87			3
88	11					1
89	89					1
90	3	5	9	15	45	5
91	7	13	91			3
92	23					1
93	3	31	93			3
94	47					1
95	5	19	95			3
96	3					1
97	97					1
98	7	49				2
99	3	9	11	33	99	5
100	5	25				2

✱ A 2.11:



The rectangular figure with the width 1 + 2 + 3 + 4 + 5 and the height 1 + 5 is composed of the squares with the areas 1^2, 2^2, 3^2, 4^2 and 5^2 and rectangular strips of the height 1 with the areas
$1 \cdot 1$, $1 \cdot (1 + 2)$, $1 \cdot (1 + 2 + 3)$, $1 \cdot (1 + 2 + 3 + 4)$ and $1 \cdot (1 + 2 + 3 + 4 + 5)$.

It therefore applies:

$(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + (1) + (1+2) + (1+2+3) + (1+2+3+4) + (1+2+3+4+5) = (1+2+3+4+5) \cdot 6$

The sums of natural numbers on the left side (in brackets) can be replaced according to the sum fomula, i.e.

$1 = \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 1$; $1 + 2 = \frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 2$; $1 + 2 + 3 = \frac{1}{2} \cdot 3^2 + \frac{1}{2} \cdot 3$; $1 + 2 + 3 + 4 = \frac{1}{2} \cdot 4^2 + \frac{1}{2} \cdot 4$;
$1 + 2 + 3 + 4 + 5 = \frac{1}{2} \cdot 5^2 + \frac{1}{2} \cdot 5$,

So we have:

$(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + (\frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 1) + (\frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 2) + (\frac{1}{2} \cdot 3^2 + \frac{1}{2} \cdot 3) + (\frac{1}{2} \cdot 4^2 + \frac{1}{2} \cdot 4)$
$+ (\frac{1}{2} \cdot 5^2 + \frac{1}{2} \cdot 5) = (1 + 2 + 3 + 4 + 5) \cdot 6$

Rearranged we get

$(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + \frac{1}{2} \cdot (1^2 + 2^2 + 3^2 + 4^2 + 5^2) + \frac{1}{2} \cdot (1 + 2 + 3 + 4 + 5) = (1 + 2 + 3 + 4 + 5) \cdot 6$

and further $\frac{3}{2} \cdot (1^2 + 2^2 + 3^2 + 4^2 + 5^2) = \frac{11}{2} \cdot (1 + 2 + 3 + 4 + 5)$

Solved to the sum of square numbers it results

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{11}{3} \cdot (1 + 2 + 3 + 4 + 5)$

Finally we replace the term with the sum of the first five consecutive numbers

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{11}{3} \cdot \frac{1}{2} \cdot (5^2 + 5) = 55$

✱ A 2.12:

For the sum of the first n square numbers we have: $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6} \cdot n \cdot (n+1) \cdot (2n+1)$

Für die Summe der ersten n geraden Quadratzahlen folgt hieraus:

$$2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = 2^2 \cdot (1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{2}{3} \cdot n \cdot (n+1) \cdot (2n+1)$$

Therefore we get for the sum of the first n odd square numbers:

$$1^2 + 3^2 + .. + (2n-1)^2 = [1^2 + 2^2 + .. + (2n)^2] - [2^2 + 4^2 \ldots + (2n)^2] = \frac{1}{6} \cdot 2n \cdot (2n+1) \cdot (4n+1) - \frac{2}{3} \cdot n \cdot (n+1) \cdot (2n+1)$$

$$= \frac{1}{3} \cdot n \cdot (2n+1) \cdot [4n + 1 - 2n - 2] = \frac{1}{3} \cdot n \cdot (2n+1) \cdot (2n-1) \cdot$$

Note: You can find wonderful illustrations under

http://www.walser-h-m.ch/hans/Miniaturen/S/Summe_unger_Quadratzahlen/Summe_unger_Quadratzahlen.htm

http://www.walser-h-m.ch/hans/Miniaturen/S/Summe_unger_Quadratzahlen2/Summe_unger_Quadratzahlen2.htm

http://www.walser-h-m.ch/hans/Miniaturen/S/Summe_unger_Quadratzahlen3/Summe_unger_Quadratzahlen3.htm

✱ A 2.13:

As the sum of two consecutiv square numbers is always an even number, they can be represented as sums of consecutive natural numbers – according to Sylvester's theorem:

Examples:

$1^2 + 2^2 = 5 = 2 + 3$;

$2^2 + 3^2 = 13 = 6 + 7$;

$3^2 + 4^2 = 25 = 5^2 = 12 + 13 = 3 + 4 + 5 + 6 + 7$;

$4^2 + 5^2 = 41 = 20 + 21$; $5^2 + 6^2 = 61 = 30 + 31$;

$6^2 + 7^2 = 85 = 5 \cdot 17 = 42 + 43 = 15 + 16 + 17 + 18 + 19 = 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13$; …

In some of these examples, there are obviously several ways to display them. However, since the examples also include prime numbers that have only *one* odd divisor (namely themselves), i.e. for which there is only *one* way of representing as a sum, there is only one common way of representation for all sums of two consecutive square numbers, namely as the sum of two consecutive numbers:

$n^2 + (n+1)^2 = n^2 + n^2 + 2n + 1 = (n^2 + n) + (n^2 + n + 1) = [\, n \cdot (n+1) \,] + [\, n \cdot (n+1) + 1 \,]$

✱ A 2.14:

Each of the triangles on the left side contains once the number 1, twice the number 2, three times the number 3, four times the number 4, five times the number 5, six times the number 6 and seven times the number 7; the sum of all fields of a triangle is therefore

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$.

In the triangle on the right, each cell contains the sum $2 \cdot 7 + 1 = 15$.

Therefor ewe have

$3 \cdot (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2) = (1 + 2 + 3 + 4 + 5 + 6 + 7) \cdot (2 \cdot 7 + 1) = 420$, i. e.

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 140$.

Generally we have

$3 \cdot (1^2 + 2^2 + 3^2 + \ldots + n^2) = (1 + 2 + 3 + \ldots + n) \cdot (2n + 1)$ according to

$1 + 2 + 3 + \ldots + n = \frac{1}{2} \cdot n \cdot (n + 1)$ we get the proposition.

✱ A 2.15:

We can find the following properties:

– The sum of the first 32 odd cube numbers is equal to the sum of 11 consecutive powers of 2 (starting with the exponent 10):
  $1^3 + 3^3 + .. + 63^3 = 2^{10} + 2^{11} + \ldots + 2^{20} = 111111111110000000000_2$.

– The sum of the first 64 odd cube numbers is equal to the sum of 13 consecutive powers of 2 (starting with the exponent 12):
  $1^3 + 3^3 + .. + 127^3 = 2^{12} + 2^{11} + \ldots + 2^{24} = 1111111111111000000000000_2$.

From formula 2.5 we get a formula for the sum of the first n even cube numbers:

$$2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = 2^3 \cdot (1^3 + 2^3 + 3^3 + \ldots + n^3) = 2 \cdot n^2 \cdot (n+1)^2$$

Therefore it results for the sum of the first n odd cube numbers:

$$1^3 + 3^3 + .. + (2n-1)^3 = [1^3 + 2^3 + .. + (2n)^3] - [2^3 + 4^3 \ldots + (2n)^3] = \tfrac{1}{4} \cdot (2n)^2 \cdot (2n+1)^2 - 2 \cdot n^2 \cdot (n+1)^2$$

$$= n^2 \cdot (2n+1)^2 - 2 \cdot n^2 \cdot (n+1)^2 = n^2 \cdot (4n^2 + 4n + 1 - 2n^2 - 4n - 2) = n^2 \cdot (2n^2 - 1) = 2n^4 - n^2 \text{, i. e.}$$

$$1^3 + 3^3 + .. + (2n-1)^3 = 2n^4 - n^2 .$$

In the case that the odd number 2n – 1 is the predecessor of a power of two, i. e. for

2 2 · 2 – 1 = 3, 2 · 2² – 1 = 7, 2 · 2³ – 1 = 15, 2 · 2⁴ – 1 = 31 and so on,

you can also note this relationship as follows

$$1^3 + 3^3 + .. + (2 \cdot 2^k - 1)^3 = 2 \cdot (2^k)^4 - (2^k)^2 = 2^{4k+1} - 2^{2k}.$$

Powers of two have the special property that they are each greater by 1 than the sum of all powers of two with a smaller exponent: $1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1$.

Therefore we kann write: $2^{4k+1} = \left(1 + 2 + 2^2 + 2^3 + \ldots + 2^{4k}\right) + 1$ und $2^{2k} = \left(1 + 2 + 2^2 + 2^3 + \ldots + 2^{2k-1}\right) + 1$,

and thus $1^3 + 3^3 + .. + (2 \cdot 2^k - 1)^3 = 2^{4k+1} - 2^{2k} = 2^{2k} + 2^{2k+1} + \ldots + 2^{4k}$.

✱ A 2.16:

More examples for sums of consecutive cube numbers:

$4^3 + 5^3 = 189 = 3^3 \cdot 7 = 94 + 95 = 62 + 63 + 64 = 24 + 25 + 26 + 27 + 28 + 29 + 30$
$17 + 18 + 19 + 20 + 21 + 22 + 23 + 24 + 25 = \ldots$ (display as sum of 2, 3, 7, 9, 21, 27 or 63 numbers);

$5^3 + 6^3 = 341 = 11 \cdot 31 = 170 + 171 = 26 + 27 + 28 + 29 + 30 + 31 + 32 + 33 + 34 + 35 + 36 = \ldots$
(display as sum of 2, 11 or 31 numbers);

$6^3 + 7^3 = 559 = 13 \cdot 43 = \ldots$ (display as sum of 2, 13 or 43 numbers); …

From the examples it can be assumed that there are two ways of displaying such sums:

- Display as sum of two consecutive natural numbers:

  $n^3 + (n+1)^3 = n^3 + (n^3 + 3n^2 + 3n + 1) = [\, n^3 + ½ \cdot 3 \cdot n \cdot (n+1)\,] + [\, n^3 + ½ \cdot 3 \cdot n \cdot (n+1) + 1]$

  Note: n · (n+1) is an even number, i. e. half of it is a natural number, too.

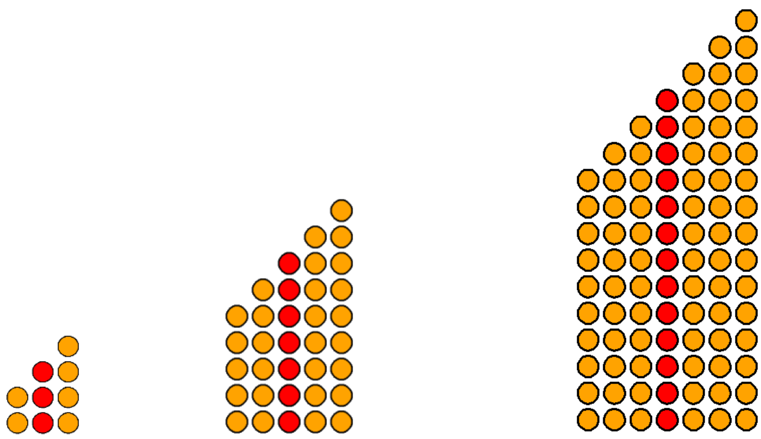
- Display as sum of 2n + 1 consecutive natural numbers: The sum of the two consecutive cube numbers $n^3 + (n+1)^3$ is divisble by the odd natural number 2n + 1:

  $n^3 + (n+1)^3 = (2n^3 + 3n^2 + 3n + 1) = (2n + 1) \cdot (n^2 + n + 1)$,

  therefore the mean summand is $n^2 + n + 1$, and thus the sum can be displayed as follows:

  $n^3 + (n+1)^3 = (n^2 + 1) + (n^2 + 2) + \ldots + (n^2 + 2n + 1) = (n^2 + 1) + (n^2 + 2) + \ldots + (n + 1)^2$

$1^3 + 2^3 = 2 + 3 + 4$ $2^3 + 3^3 = 5 + 6 + 7 + 8 + 9$ $3^3 + 4^3 = 10 + 11 + 12 + 13 + 14 + 15 + 16$



Note: A 3-dimensional illustration of this property can be found in Roger B. Nelsen: Proofs without Words II, MAA, 2000, p. 94.

✱ A 2.17:

For three adjacent L-shaped forms the following applies: An odd square number which is the sum of three consecutive odd numbers must be divisible by 3 (since the first of the three numbers is 2 less than the middle number and the third is 2 more). This results in

$9^2 = 81 = 25 + 27 + 29 = (12 + 13) + (13 + 14) + (14 + 15)$

as the total number of blue stones in the three L-shaped forms and 12^2 as the number of red stones and 15^2 as the total number of red and blue stones, see figure in the book.

Here the first number triples of the type with three L-shaped forms:

n	$a_n = 3\cdot (2n+1)$	$9 \cdot (2n + 1)^2 = 36n^2 + 36n + 9$	$b_n = 6n \cdot (n + 1)$	$c_n = 6n \cdot (n + 1) + 3$
1	9	81 = (12 + 13) + (13 + 14) + (14 + 15)	12	15
2	15	225 = (36 + 37) + (37 + 38) + (38 + 39)	36	39
3	21	441 = (72 + 73) + (73 + 74) + (74 + 75)	72	75
4	27	729 = (120 + 121) + (121 + 122) + (122 + 123)	120	123
5	33	...	180	183
…	…	...	…	…

The number triples belonging to the figures with three L-shaped forms are obtained by tripling the number of triples known from the figures with one L-shaped form:
$a_n = 2n + 1$; $b_n = 3 \cdot (2n + 1)^2$; $c_n = 2n \cdot (n + 1) + 1$

Chapter 3

✱ A 3.1:

Player No 1 wins according to game rule No 1 if the number of squares that can be drawn is even; he wins according to game rule No 2 if the number of squares of different sizes is an even number.

The following table shows an overview (which can be continued accordingly) of how the a x b rectangles (i.e. of width a and height b) can be dissected with squares as large as possible.

From the different highlighted colours you can see which player wins the game according to rule No 1 or No 2:

Player No 1 wins according to game rule No 1 and according to game rule No 2.
Player No 1 wins according to game rule No1 and looses according to game rule No 2.
Player No 1 looses according to game rule No 1 and wins according to game rule No2.
Player No 1 looses according to game rule No 1 and according to game rule No 2.

↓b a→	1	2	3	4	5	6
1	$1 \cdot 1^2$					
2	$2 \cdot 1^2$	$1 \cdot 2^2$				
3	$3 \cdot 1^2$	$1 \cdot 2^2 + 2 \cdot 1^2$	$1 \cdot 3^2$			
4	$4 \cdot 1^2$	$2 \cdot 2^2$	$1 \cdot 3^2 + 3 \cdot 1^2$	$1 \cdot 4^2$		
5	$5 \cdot 1^2$	$2 \cdot 2^2 + 1 \cdot 1^2$	$1 \cdot 3^2 + 1 \cdot 2^2 + 2 \cdot 1^2$	$1 \cdot 4^2 + 4 \cdot 1^2$	$1 \cdot 5^2$	
6	$6 \cdot 1^2$	$3 \cdot 2^2$	$2 \cdot 3^2$	$1 \cdot 4^2 + 2 \cdot 2^2$	$1 \cdot 5^2 + 5 \cdot 1^2$	$1 \cdot 6^2$

✱ A 3.2:

(1) $\frac{10}{3} = 3 + \frac{1}{3} = [3\,;3]$ (2) $\frac{13}{4} = 3 + \frac{1}{4} = [3\,;4]$ (3) $\frac{11}{5} = 2 + \frac{1}{5} = [2\,;5]$

(4) $\frac{17}{6} = 2 + \frac{5}{6} = 2 + \frac{1}{\frac{6}{5}} = 2 + \frac{1}{1+\frac{1}{5}} = [2\,;1,5]$

(5) $\frac{16}{7} = 2 + \frac{2}{7} = 2 + \frac{1}{\frac{7}{2}} = 2 + \frac{1}{3+\frac{1}{2}} = [2\,;3,2]$

(6) $\frac{19}{8} = 2 + \frac{3}{8} = 2 + \frac{1}{\frac{8}{3}} = 2 + \frac{1}{2+\frac{2}{3}} = 2 + \frac{1}{2+\frac{1}{\frac{3}{2}}} = 2 + \frac{1}{2+\frac{1}{1+\frac{1}{2}}} = [2\,;2,1,2]$

✱ A 3.3:

$$[a_0\,;a_1\,,a_2\,,a_3] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}} = a_0 + \cfrac{1}{\cfrac{a_1 \cdot a_2 \cdot a_3 + a_1 + a_3}{a_2 \cdot a_3 + 1}}$$

$$= \frac{a_0 \cdot a_1 \cdot a_2 \cdot a_3 + a_0 \cdot a_1 + a_0 \cdot a_3 + a_2 \cdot a_3 + 1}{a_1 \cdot a_2 \cdot a_3 + a_1 + a_3}$$

✱ A 3.4:

For example, if in the continued fraction [a_0 ; a_1 , a_2 , a_3] the following holds for a_3: a_3 = 1, then the last mixed fraction in the continued fraction is equal to $a_2 + \frac{1}{a_3}$, i.e. a_2 + 1, the last mixed fraction does not contain a fraction at all, but is a natural number.

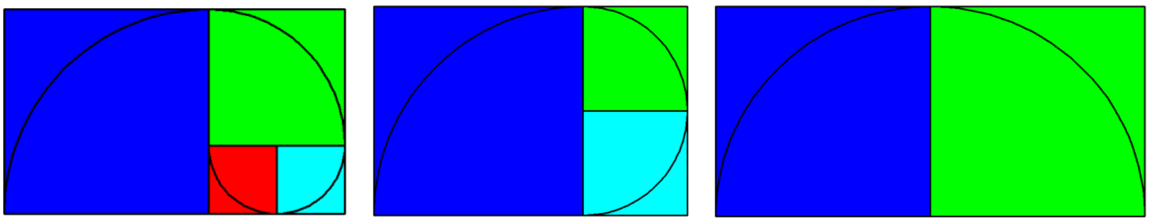
Therefore we have: [a_0 ; a_1 , a_2, …, a_{n-1}, 1] = [a_0 ; a_1 , a_2, …, a_{n-1} + 1]

✱ A 3.5:

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}} = 1 + \cfrac{1}{2 + \cfrac{1}{\cfrac{13}{4}}} = 1 + \cfrac{1}{2 + \cfrac{4}{13}} = 1 + \cfrac{1}{\cfrac{30}{13}} = 1 + \frac{13}{30} = \frac{43}{30}$$

✱ A 3.6:

Dissection of a 3x5-rectangle, a 2x3-rectangle and a 1x2-rectangle with a spiral consisting of quarter circles



✱ A 3.7:

From (3.1) we get:

$$[1;2\,,n] = \frac{1 \cdot 2 \cdot n + 1 + n}{2 \cdot n + 1} = \frac{3n+1}{2n+1} = \frac{3 + \frac{1}{n}}{2 + \frac{1}{n}} \to \frac{3}{2},$$

i. e. the ratio of the side lengths of the rectangle converges to 1.5, so the ratio of the side lengths converge to 3 : 2.

From (3.2) we get:

$$[1;1,2\,,n] = \frac{1 \cdot 1 \cdot 2 \cdot n + 1 \cdot 1 + 1 \cdot n + 2 \cdot n + 1}{1 \cdot 2 \cdot n + 1 + n} = \frac{5n+2}{3n+1} = \frac{5 + \frac{2}{n}}{3 + \frac{1}{2}} \to \frac{5}{3}$$

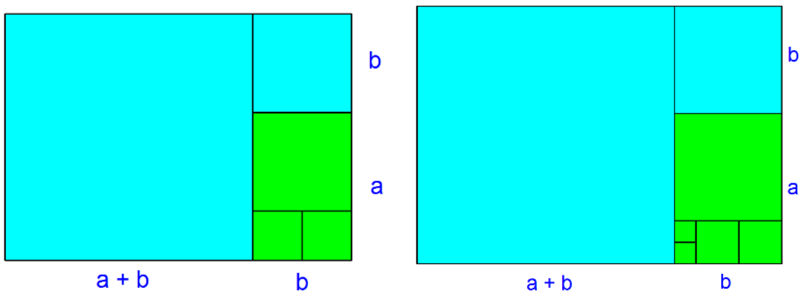
✱ A 3.8:

Ratio of the side lengths	$\frac{5}{2} = 2.5$	$\frac{8}{3} = 2.\overline{6}$	$\frac{13}{5} = 2.6$	$\frac{21}{8} = 2.625$	$\frac{34}{13} = 2.61538...$	…
Dissection	[2 ; 2]	[2 ; 1 , 2]	[2 ; 1 , 1 , 2]	[2 ; 1 , 1 , 1 , 2]	[2 ; 1 , 1 , 1 , 1 , 2]	…

This sequence has the same values as the sequence of quotients of consecutive Fibonacci numbers, augmented by 1, as can generally be demonstrated as follows

$$\frac{f_{n+1}}{f_{n-1}} = \frac{f_n + f_{n-1}}{f_{n-1}} = \frac{f_n}{f_{n-1}} + 1$$

✱ A 3.9:



The first quotient describes a rectangle with the side lengths a and b (the rectangle for the continued fraction [1; 2] is colored green). When changing to the continued fraction [1; 2, 2] this rectangle has to be supplemented by a square with the side length b and by a square with the side length a + b, so that a rectangle with side lengths (a + b) + b and a + b arises.
This also applies to the next steps, see figure on the left.

✱ A 3.10:

Ratio of side lengths	$\frac{5}{3} = 1.\overline{6}$	$\frac{19}{11} = 1.\overline{72}$	$\frac{71}{41} = 1.\overline{73170}$	…
Dissection	[1 ; 1 , 2]	[1 ; 1 , 2 , 1 , 2]	[1 ; 1 , 2 , 1 , 2 , 1 , 2]	…
Continued fraction	$1+\cfrac{1}{1+\cfrac{1}{2}}$	$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}}$	$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}}}}$	…

The elements of this sequence of quotients increase more and more slowly and converge to a limit:

If you start with $\frac{a}{b}$ for the first quotient, then the next quotient is $\frac{2a+3b}{a+2b}$.

Therefore the following applies for the limit:

$$\frac{a}{b} = \frac{2a+3b}{a+2b} \Leftrightarrow a \cdot (a+2b) = (2a+3b) \cdot b \Leftrightarrow a^2 + 2ab = 2ab + 3b^2 \Leftrightarrow a^2 = 3b^2 \Leftrightarrow \left(\frac{a}{b}\right)^2 = 3$$

Thus the limit is $[1;\overline{1,2}]=\sqrt{3}$. The rectangles associated to this sequence converge to a rectangle with the ratio of side $\sqrt{3}:1$ length.

✱ A 3.11:

Ratio of side lengths	$\frac{3}{2}=1.5$	$\frac{7}{5}=1.4$	$\frac{17}{12}=1.41\overline{6}$	$\frac{41}{29}=1.4137...$	$\frac{99}{70}=1.414285...$
Difference to $\sqrt{2}$	+0.085786	-0.014214	+0.002453	-0.000421	+0.000072

Ratio of side lengths	$\frac{5}{3}=1.\overline{6}$	$\frac{19}{11}=1.\overline{72}$	$\frac{71}{41}=1.\overline{73170}$	$\frac{265}{153}=1.732026...$	$\frac{989}{571}=1.732049...$
Difference to $\sqrt{3}$	-0.065384	-0.004778	-0.000343	-0.00025	-0.000002

✱ A 3.12:

$$\sqrt{6}=2+\left(\sqrt{6}-2\right)=2+\frac{\left(\sqrt{6}-2\right)\cdot\left(\sqrt{6}+2\right)}{\left(\sqrt{6}+2\right)}=2+\frac{2}{2+\sqrt{6}}=2+\frac{2}{2+2+\frac{2}{2+\sqrt{6}}}$$

$$=2+\frac{1}{2+\frac{1}{2+\sqrt{6}}}=2+\frac{1}{2+\frac{1}{2+2+\frac{1}{2+\frac{1}{2+\sqrt{6}}}}}$$

$$=2+\frac{1}{2+\frac{1}{4+\frac{1}{2+\frac{1}{2+2+\frac{1}{2+\frac{1}{2+\sqrt{6}}}}}}}=[\,2;\overline{2,4}\,]$$

$$\sqrt{7}=2+\left(\sqrt{7}-2\right)=2+\frac{\left(\sqrt{7}-2\right)\cdot\left(\sqrt{7}+2\right)}{\left(\sqrt{7}+2\right)}=2+\frac{3}{2+\sqrt{7}}=2+\frac{1}{\frac{2+\sqrt{7}}{3}}=2+\frac{1}{1+\frac{\sqrt{7}-1}{3}}$$

$$=2+\frac{1}{1+\frac{6}{3\cdot\left(\sqrt{7}+1\right)}}=2+\frac{1}{1+\frac{1}{\frac{\sqrt{7}+1}{2}}}=2+\frac{1}{1+\frac{1}{1+\frac{\sqrt{7}-1}{2}}}=2+\frac{1}{1+\frac{1}{1+\frac{6}{2\cdot\left(\sqrt{7}+1\right)}}}$$

$$=2+\frac{1}{1+\frac{1}{1+\frac{1}{\frac{\sqrt{7}+1}{3}}}}=2+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{\sqrt{7}-2}{3}}}}=2+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\sqrt{7}}}}}$$

$$=2+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+2+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\sqrt{7}}}}}}}}}=\ldots=[\,2\,;\overline{1,1,1,4}\,]$$

✱ A 3.13:

The reciprocal of the continued fraction [a; b], i. e. $a+\frac{1}{b}$, is $\frac{1}{a+\frac{1}{b}}=0+\frac{1}{a+\frac{1}{b}}$, i. e. [0; a, b].

Generally applies: [0; a_1, a_2, a_3, …, a_n] and [a_1; a_2, a_3, …, a_n] are inverse to each other.

✱ A 3.14:

After the first elements 4/1 = [4;] ; 9/4 = [2; 4] ; 16/9 = [1; 1, 3, 2] you can realize that there is a certain rule:

25/16 = [1; 1, 1, 3, 2]	36/25 = [1; 2, 3, 1, 2]
49/36 = [1; 2, 1, 3, 3]	64/49 = [1; 3, 3, 1, 3]
81/64 = [1; 3, 1, 3, 4]	100/81 = [1; 4, 3, 1, 4]
121/100 = [1; 4, 1, 3, 5]	144/121 = [1; 5, 3, 1, 5]
169/144 = [1; 5, 1, 3, 6]	196/169 = [1; 6, 3, 1, 6]

generally we have:

$(2n + 1)^2/(2n)^2$ = [1; n – 1, 1, 3, n]　　　$(2n)^2/(2n – 1)^2$ = [1; n – 1, 3, 1, n – 1]

Chapter 4

✱ A 4.1:

- 3 possible combinations for the area 14π:

$1\pi + 13\pi = 3\pi + 11\pi = 5\pi + 9\pi = 14\pi$

- 4 possible combinations for the area 16π and for 18π:

$1\pi + 15\pi = 3\pi + 13\pi = 5\pi + 11\pi = 7\pi + 9\pi = 16\pi$

$1\pi + 17\pi = 3\pi + 15\pi = 5\pi + 13\pi = 7\pi + 11\pi = 18\pi$

- 5 possible combinations for the area 20π and for 22π:

$1\pi + 19\pi = 3\pi + 17\pi = 5\pi + 15\pi = 7\pi + 13\pi = 9\pi + 11\pi = 20\pi$

$1\pi + 21\pi = 3\pi + 19\pi = 5\pi + 17\pi = 7\pi + 15\pi = 9\pi + 13\pi = 22\pi$

allgemein:

- n possible combinations for the area $4n \cdot \pi$ and for $(4n + 2) \cdot \pi$:

$1\pi + (4n - 1) \cdot \pi,\ 3\pi + (4n - 3) \cdot \pi,\ 5\pi + (4n - 5) \cdot \pi,\ \ldots,\ (2n - 1) \cdot \pi + (2n + 1) \cdot \pi$

$1\pi + (4n + 1) \cdot \pi,\ 3\pi + (4n - 1) \cdot \pi,\ 5\pi + (4n - 3) \cdot \pi,\ \ldots,\ (2n - 1) \cdot \pi + (2n + 3) \cdot \pi$

✱ A 4.2:

The following table contains all possible combinations of three odd summands up to the maximum sum of 39π: in the column on the left, you see the sums of two summands, and in the row on the top, the third summands.

Thus there are

1 possibility for the sum 9π of three circular rings (namely $1\pi + 3\pi + 5\pi$),

1 possibility for the sum 11π …

2 possibilities for the sum 13π …

3 possibilities for the sum 15π …

4 possibilities for the sum 17π …

5 possibilities for the sum 19π …

7 possibilities for the sum 21π …

8 possibilities for the sum 23π …

10 possibilities for the sum 25π …

12 possibilities for the sum 27π …

14 possibilities for the sum 29π …

16 possibilities for the sum 31π …

19 possibilities for the sum 33π …

21 possibilities for the sum 35π …

24 possibilities for the sum 37π …

27 possibilities for the sum 39π of three circular rings

The website https://oeis.org/ can help to find the next elements of the sequence.

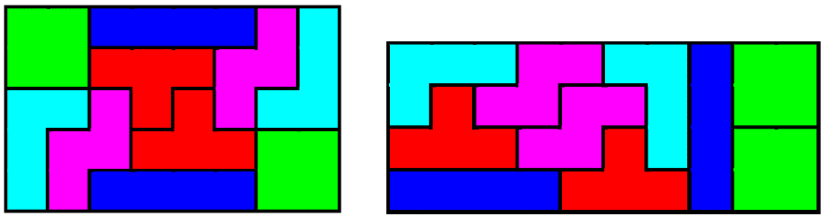
Chapter 5

✱ A 5.1:

The element “T” covers three dark fields and one light field (or vice versa). The rectangle has the same number of light fields as dark fields in a checkerboard coloring. No matter how you color the squares of the five tetrominoes: Eleven squares have one color and nine squares have the other color – thus it does not match.

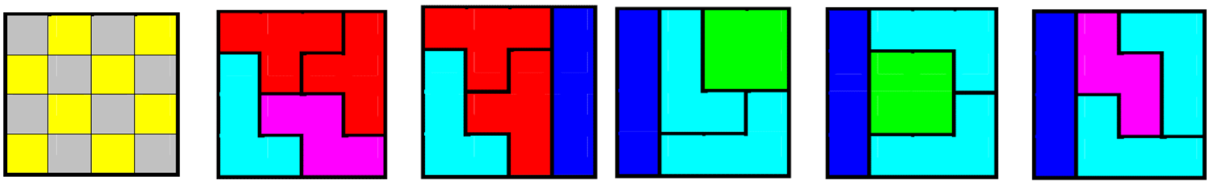
✱ A 5.2:

It is not possible to tessellate a 2x20 rectangle.



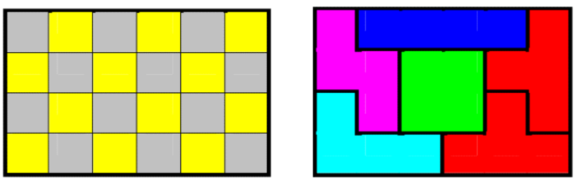
✱ A 5.3:

If you use the element “T”, you must use a second time (because of the chessboard coloring), and if it is not used, you will quickly see that the other four elements do not fit into the square at the same time.



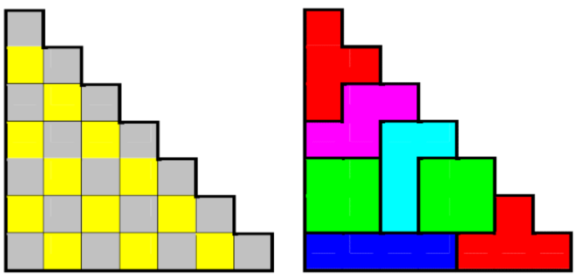
✱ A 5.4:

To lay out a rectangle with 4x6 = 24 squares with at least one element of each type means to use one piece of type I, O, L or S and two pieces of type T, which must be placed in such a way that 3 dark squares and 1 light square are covered by one of the two “T”s and 1 dark square and 3 light squares by the other “T”.



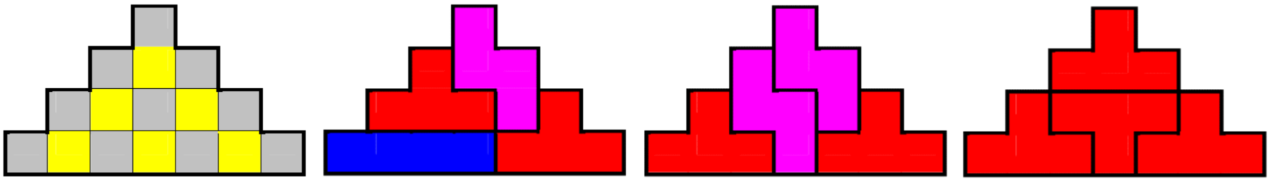
✱ A 5.5:

If you colour the squares in the style of a chessboard, 16 squares are colored in one color and 12 in the other. This means that for the tessellation with tetrominoes, the element T must be used exactly twice.

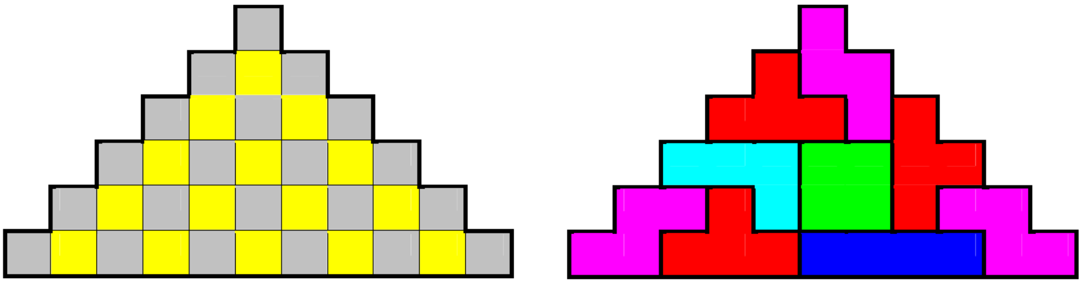


✱ A 5.6:

16 squares: If you use pieces of type T when laying the triangular figure shown with 1 + 3 + 5 + 7 = 16 squares, then you need 2 or 4 of them, because in a chessboard coloring there must be 1 + 2 + 3 = 6 colored in one color and the remaining 10 in the other color. Pieces of type L or type O cannot be used, as they would block out individual units into which no tetromino would fit in.

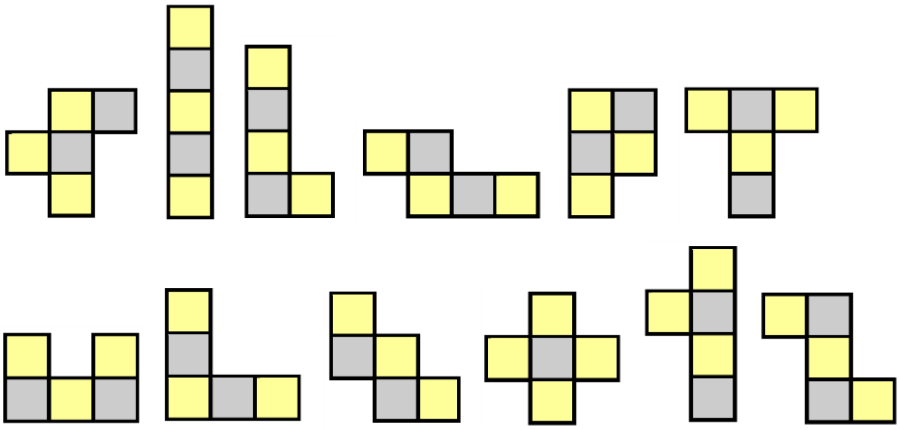


36 squares: Considering the chessboard coloring, 1 + 2 + 3 + 4 + 5 = 15 squares would be colored in one color and the remaining 21 in the other. Because of this difference 3 (or 5 or 7) pieces of type T are necessary. Since the figure is large enough, pieces of type L or type O can be used, too.



✱ A 5.7:

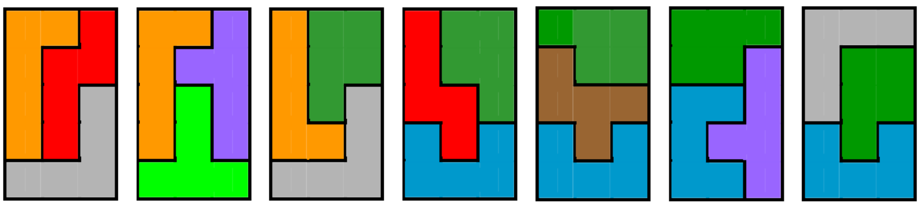
With the exception of element T, where 4 squares are colored in one color and 1 square in the other color, the ratio is 3 to 2 for all the other types.



✱ A 5.8:

All possibilities can be found on the websites listed in the “references” (see ch. 5.4).

✱ A 5.9:



✱ A 5.10:

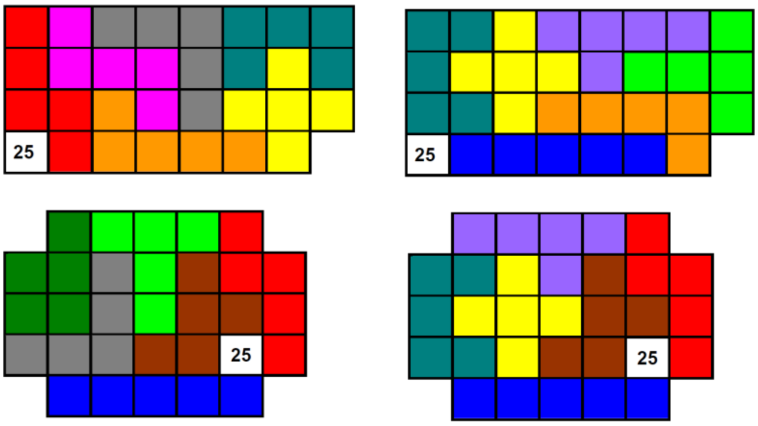
All possibilities can be found on the websites listed in the “references” (see ch. 5.4).

✱ A 5.11:

All possibilities can be found on the websites listed in the “references” (see ch. 5.4).

✱ A 5.12:

Here are two examples for the 25th of a month.



✱ A 5.13:

All possibilities can be found on the websites listed in the “references” (see ch. 5.4).

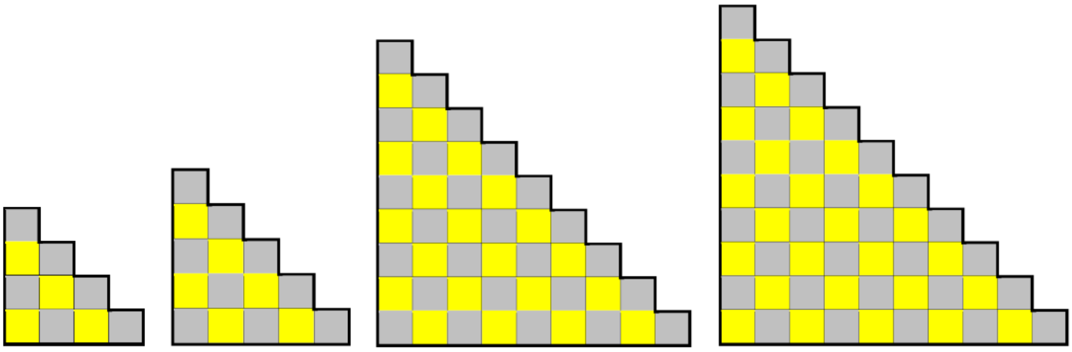
✱ A 5.14:

In the triangular figure with 1 + 2 + 3 + 4 = 10 squares you have 1 + 3 = 4 squares in one color and 2 + 4 = 6 in the other color,

in the triangular figure with 1 + 2 + 3 + 4 + 5 = 15 squares you have 2 + 4 = 6 in one color and 1 + 3 + 5 = 9 in the other color,

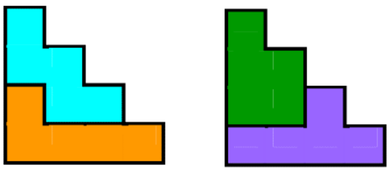
in the triangular figure with 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45 squares you have 2 + 4 + 6 + 8 = 20 in one color and 1 + 3 + 5 + 7 + 9 = 25 in the other color,

in the triangular figure with 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55 squares you have 2 + 4 + 6 + 8 + 10 = 30 in one color and 1 + 3 + 5 + 7 + 9 = 25 in the other color.



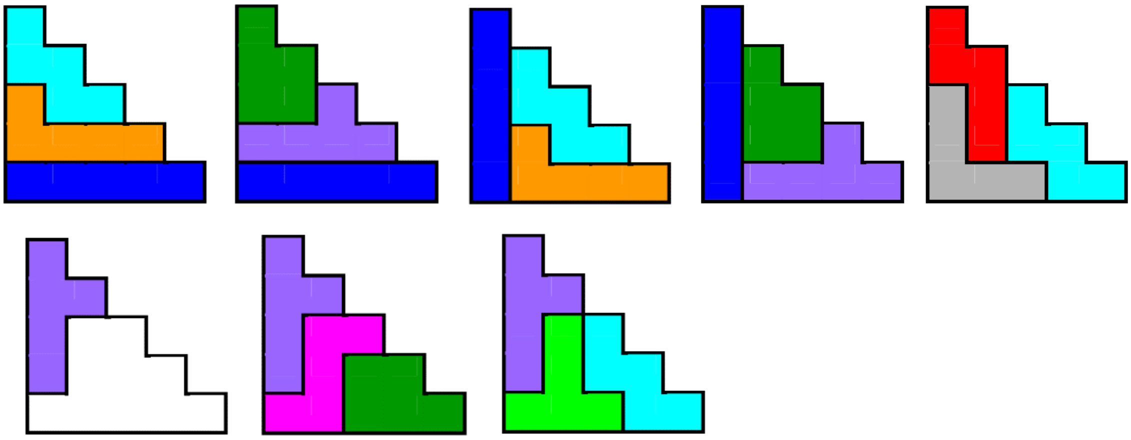
✱ A 5.15:

There are two possibilities to tessellate the triangular figure of 10 squares with two pentominoes.



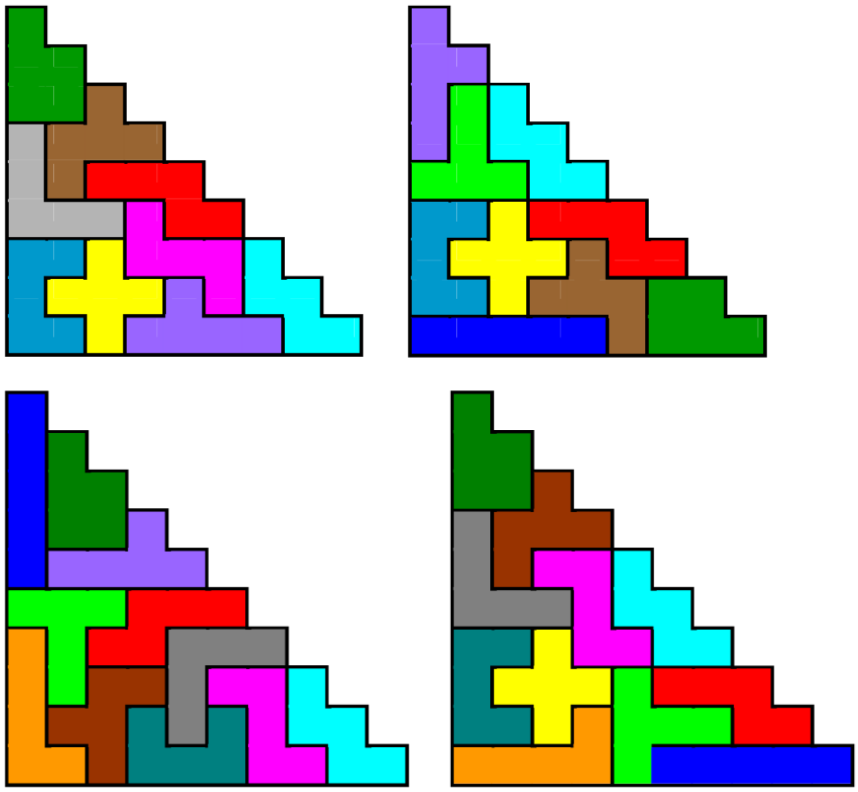
There are seven possibilities to tessellate the triangular figure of 15 squares with three pentominoes.

In the first four cases, the tessellation of the triangular figures with 10 squares is supplemented by a pentomino of type I. A fifth type is shown on the right. The last two possibilities can be found if you start with type Y (see picture on the right where the remaining squares are left white).

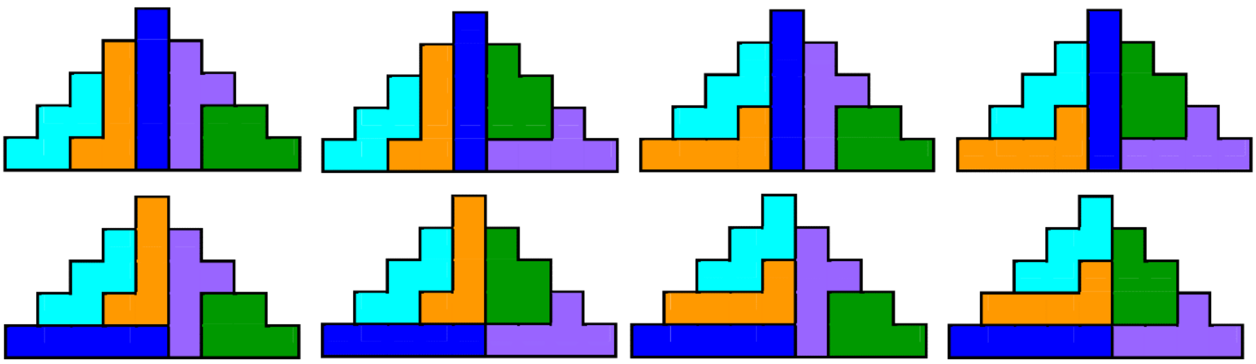


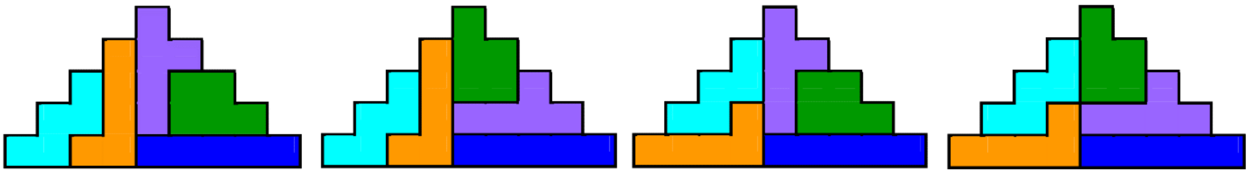
✱ A 5.16:

Here are two examples each.

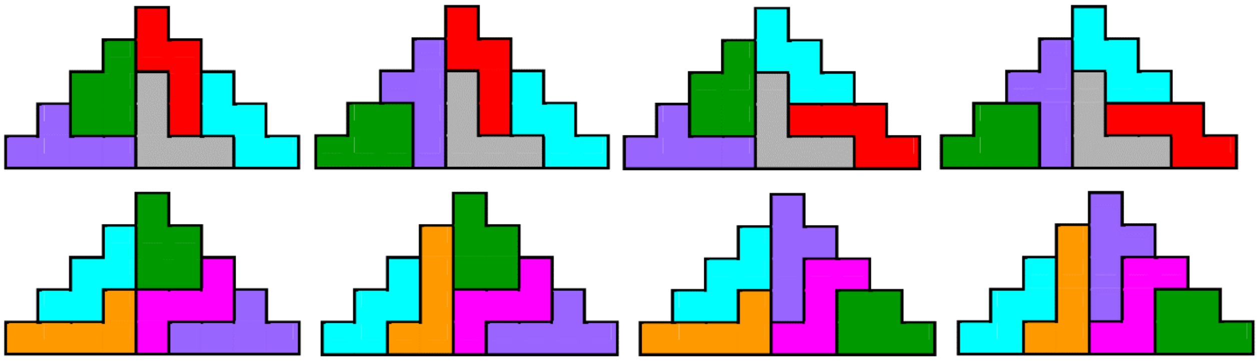


✱ A 5.17:



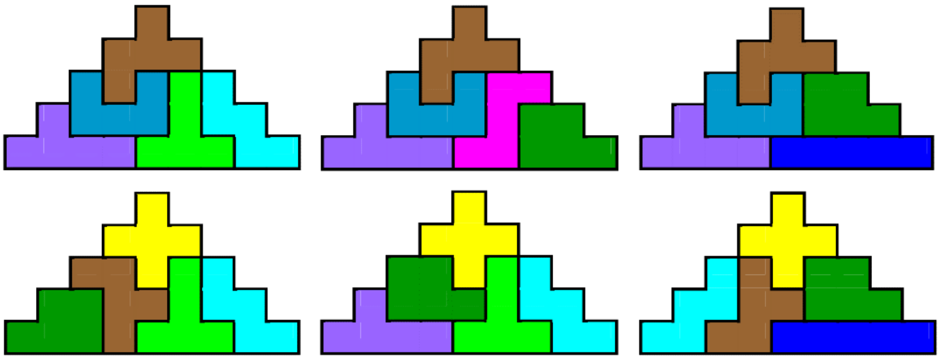


In the following picture, you see how we used the possibilities from above (how to tessellate triangles with 10 or 15 squares).

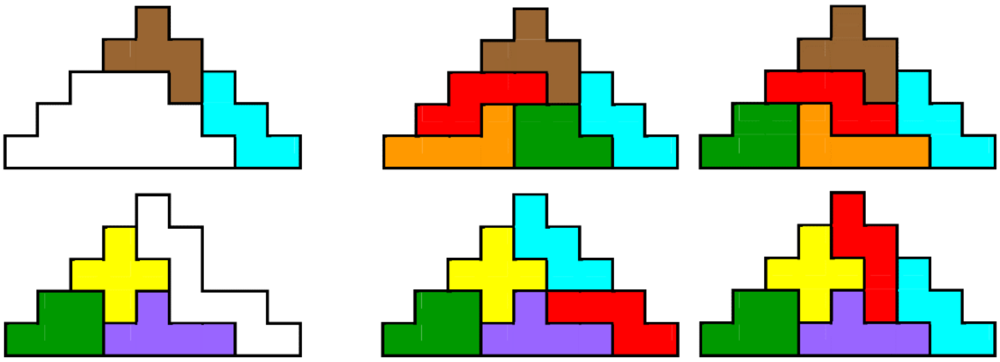


In the following illustrations, we show how partial areas can be tessellated in different ways.

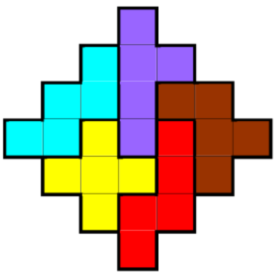
There are four different possibilities for the partial area on the left and three possibilities for the partial area on the right. However, these cannot be combined independently of each other.



In the following solutions we use the symmetrical situation.



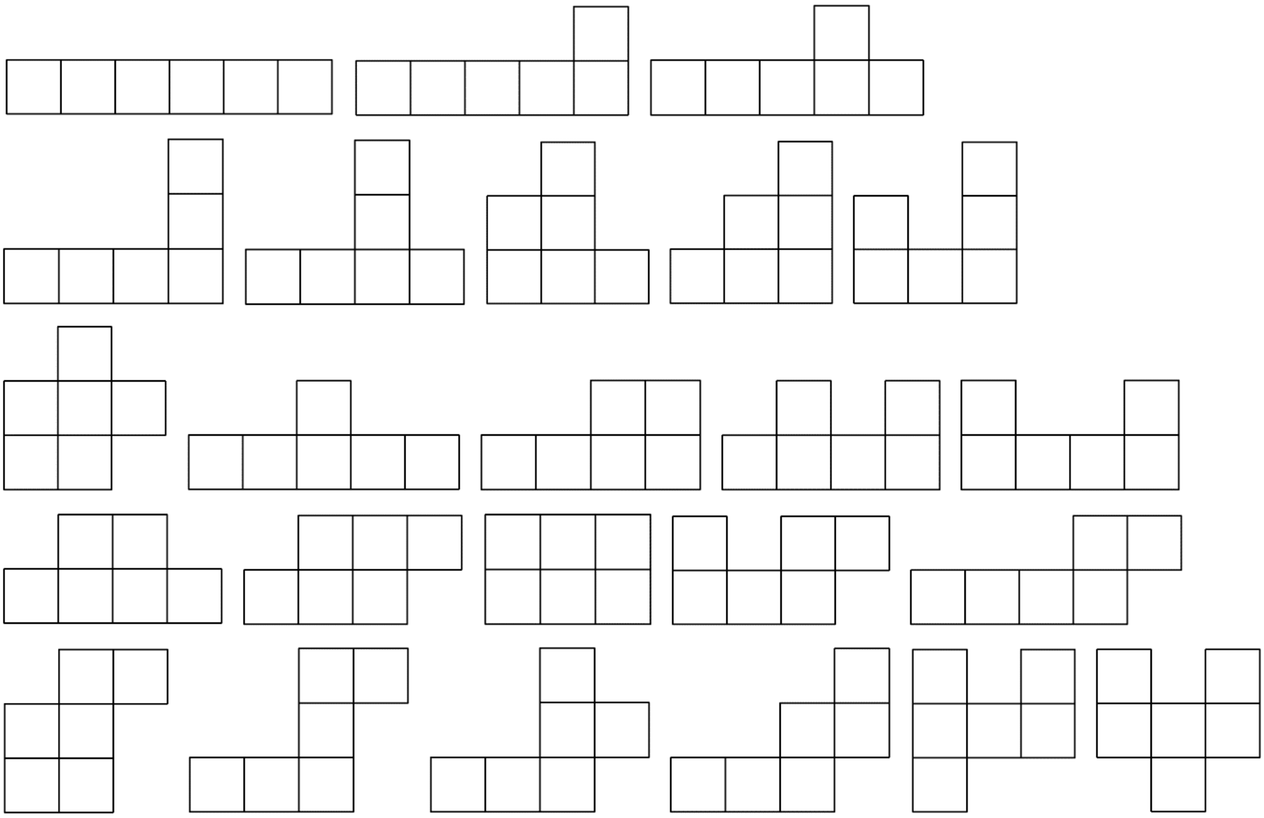
✱ A 5.18:



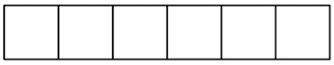
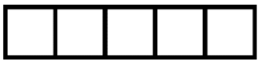
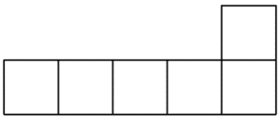
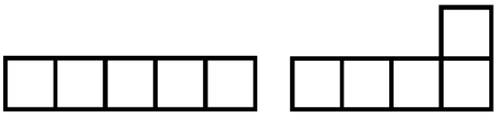
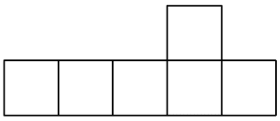
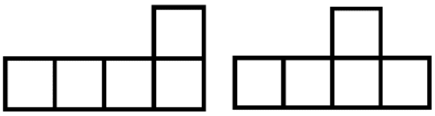
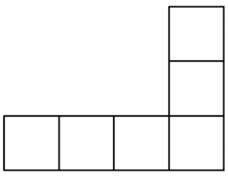
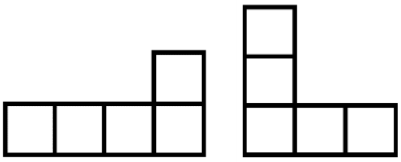
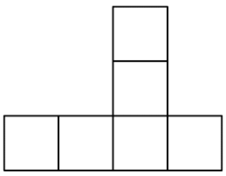
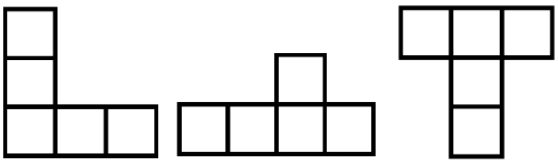
Have you found other solutions? (→ strick.lev@t-online.de)

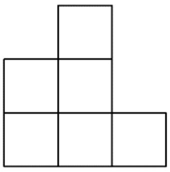
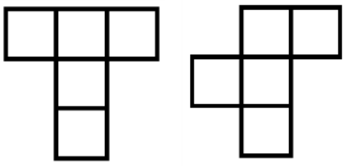
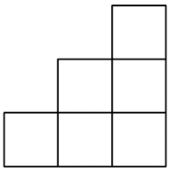
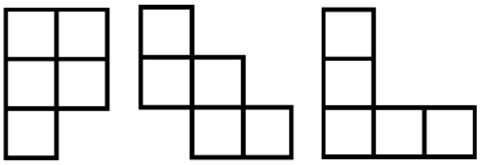
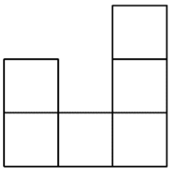
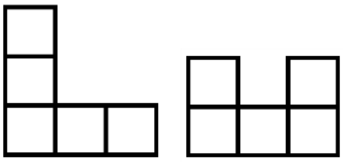
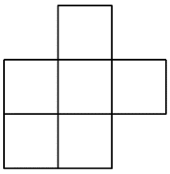
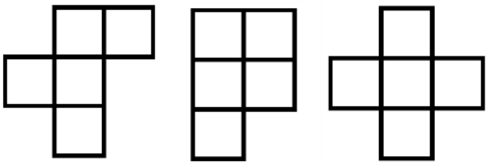
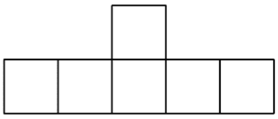
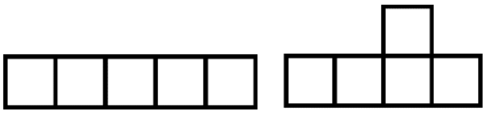
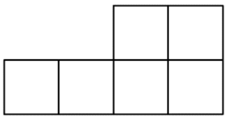
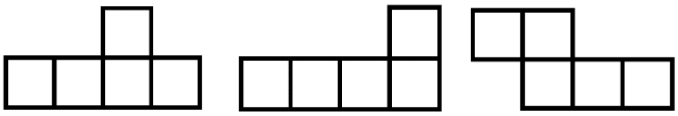
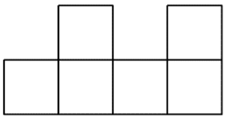
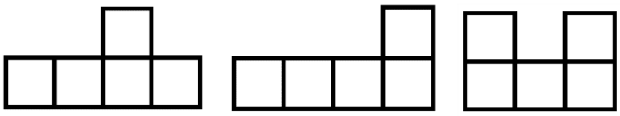
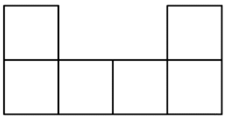
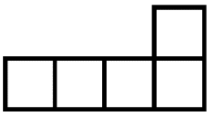
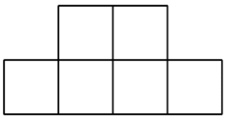
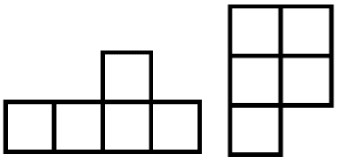
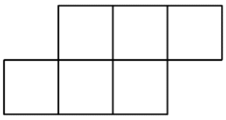
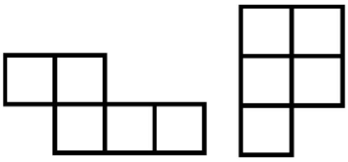
Heinz Klaus Strick: Mathematics is beautiful, Springer, ISBN: 978-3-662-59060-7

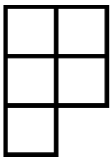
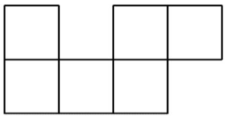
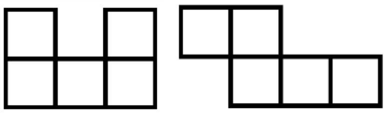
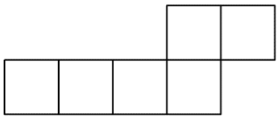
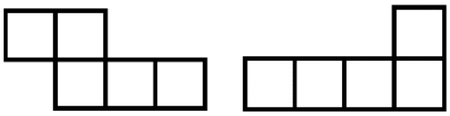
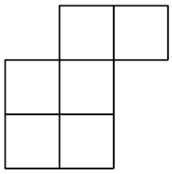
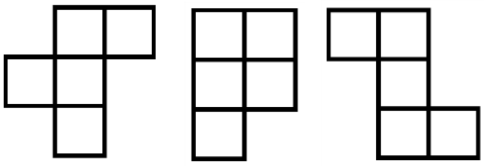
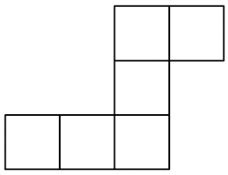
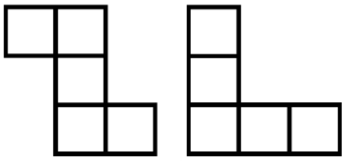
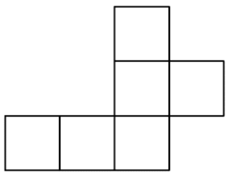
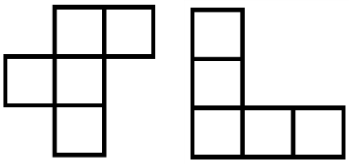
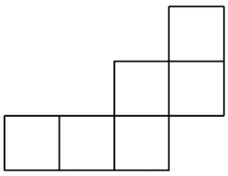
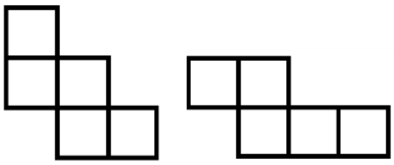
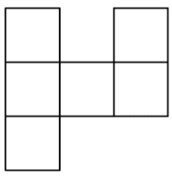
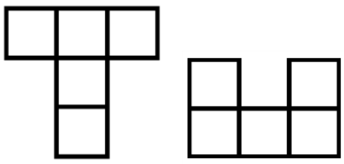
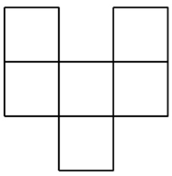
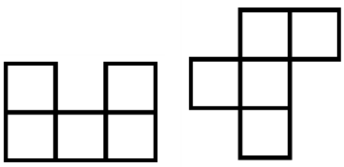
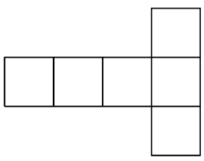
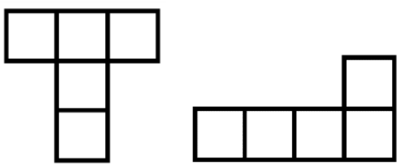
✱ A 5.19:

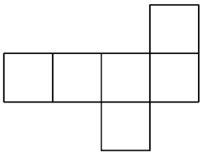
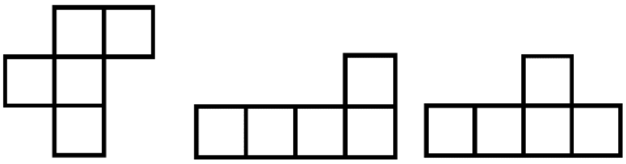
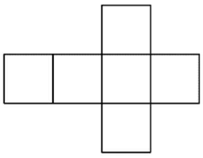
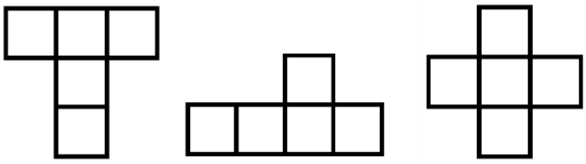
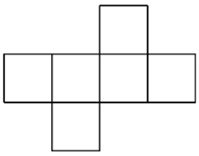
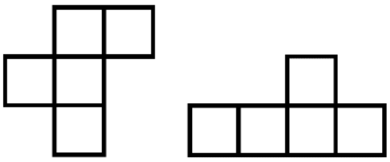
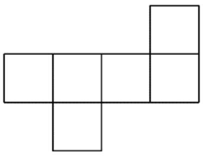
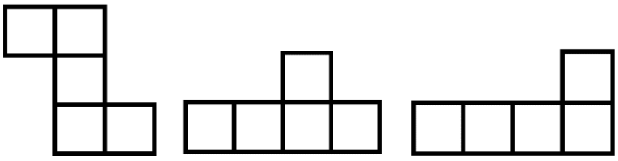
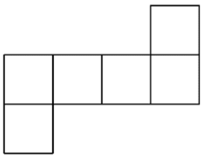
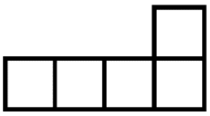
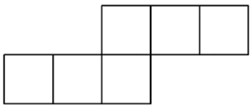
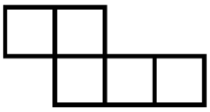
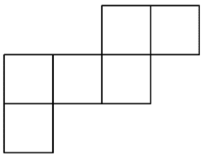
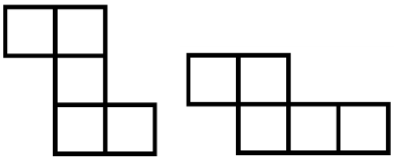
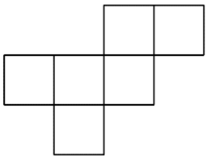
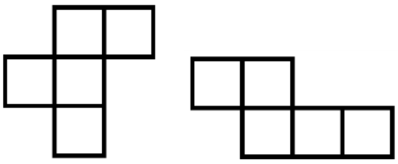
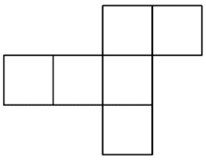
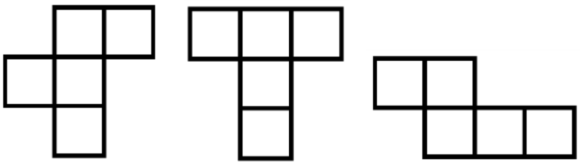


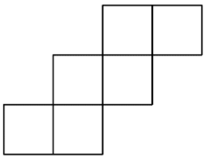
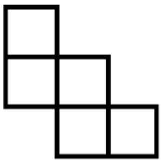
✱ A 5.20:

Nr.	hexominoes	initial pentominoes
1		
2		
3		
4		
5		

6		
7		
8		
9		
10		
11		
12		
13		
14		
15		

16		
17		
18		
19		
20		
21		
22		
23		
24		
25		

26		
27		
28		
29		
30		
31		
32		
33		
34		

35		

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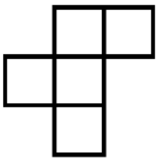
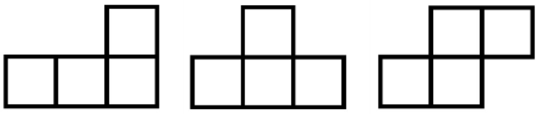
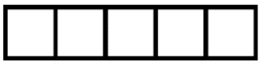
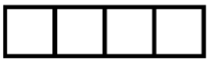
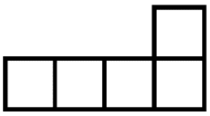
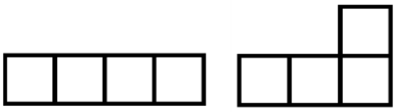
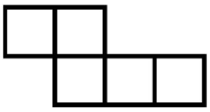
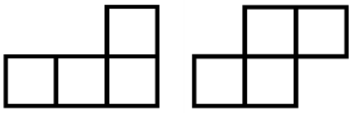
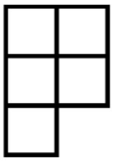
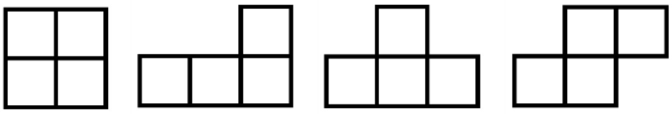
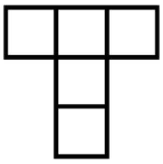
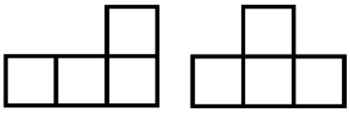
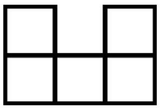
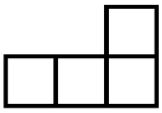
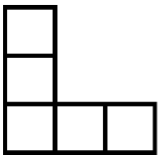
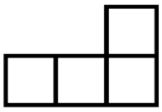
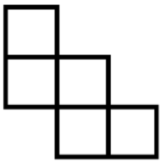
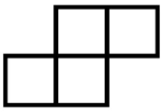
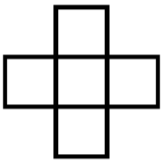
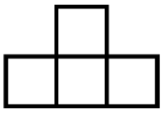
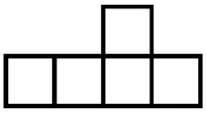
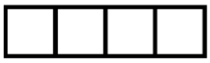
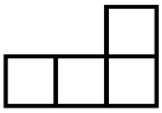
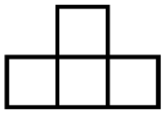
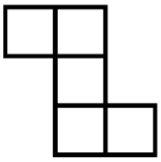
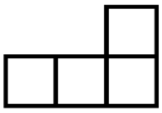
1		
2		
3		
4		
5		
6		
7		
8		
9		

10		
11		  
12		

	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35
1 + 3	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
1 + 5	x	13	15	17	19	21	23	25	27	29	31	33	35	37	39	
1 + 7	x	x	17	19	21	23	25	27	29	31	33	35	37	39		
1 + 9	x	x	x	21	23	25	27	29	31	33	35	37	39			
1 + 11	x	x	x	x	25	27	29	31	33	35	37	39				
1 + 13	x	x	x	x	x	29	31	33	35	37	39					
1 + 15	x	x	x	x	x	x	33	35	37	39						
1 + 17	x	x	x	x	x	x	x	37	39							
3 + 5	x	15	17	19	21	23	25	27	29	31	33	35	37	39		
3 + 7	x	x	19	21	23	25	27	29	31	33	35	37	39			
3 + 9	x	x	x	23	25	27	29	31	33	35	37	39				
3 + 11	x	x	x	x	27	29	31	33	35	37	39					
3 + 13	x	x	x	x	x	31	33	35	37	39						
3 + 15	x	x	x	x	x	x	35	37	39							
3 + 17	x	x	x	x	x	x	x	39								
5 + 7	x	x	21	23	25	27	29	31	33	35	37	39				
5 + 9	x	x	x	25	27	29	31	33	35	37	39					
5 + 11	x	x	x	x	29	31	33	35	37	39						
5 + 13	x	x	x	x	x	33	35	37	39							
5 + 15	x	x	x	x	x	x	37	39								
7 + 9	x	x	x	27	29	31	33	35	37	39						
7 + 11	x	x	x	x	31	33	35	37	39							
7 + 13	x	x	x	x	x	35	37	39								
7 + 15	x	x	x	x	x	x	39									
9 + 11	x	x	x	x	33	35	37	39								
9 + 13	x	x	x	x	x	37	39									
11 + 13	x	x	x	x	x	39										

Another way to find the number of combinations could be done using a spreadsheet as follows:

You start by noting all dual numbers whose sum cross is 3. With these dual numbers you can generate vectors whose components consist of 3 ones and otherwise zeros. Then the dot product is formed with a vector of the same dimension whose components are odd numbers in ascending order (1, 3, 5, 7, 9, ...).

The digits of the dual numbers are noted in reverse order.

By this method you get all sums with three odd summands.

Examples:

$7_{10} = {}_{2}111$: (1,1,1) * (1, 3, 5) = 9

$11_{10} = {}_{2}1011$: (1,1,0,1) * (1, 3, 5, 7) = 11

$13_{10} = {}_{2}1101$: (1,0,1,1) * (1, 3, 5, 7) = 13

$14_{10} = {}_{2}1110$: (0,1,1,1) * (1, 3, 5, 7) = 15

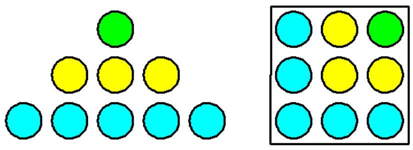
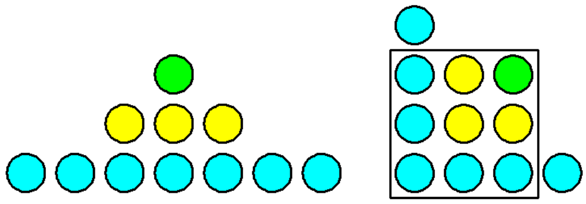
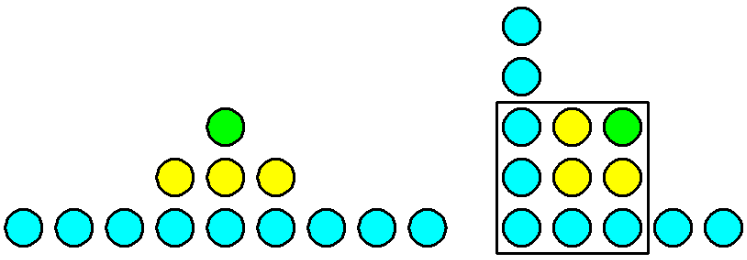
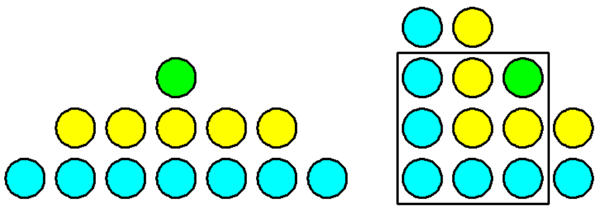
$19_{10} = {}_{2}10011$: (1,1,0,0,1) * (1, 3, 5, 7, 9) = 13

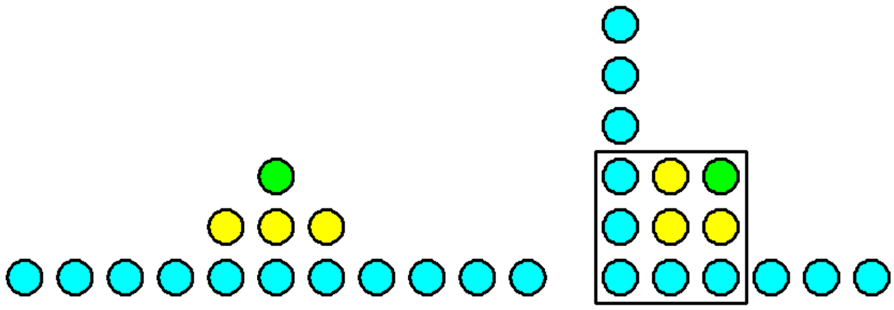
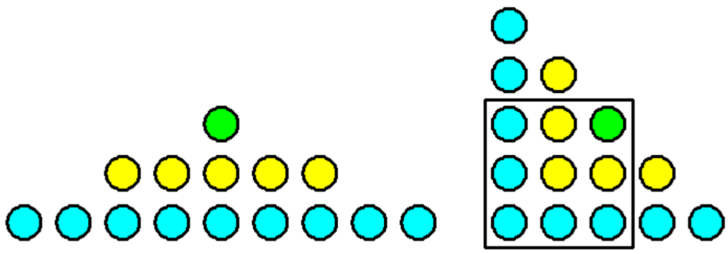
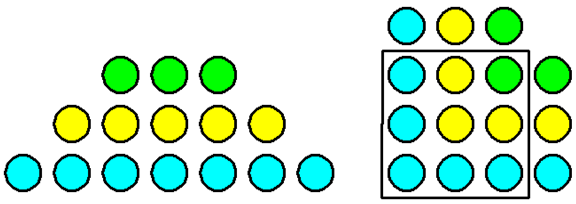
$21_{10} = {}_{2}10101$: (1,0,1,0,1) * (1, 3, 5, 7, 9) = 15

etc.

Then you (the spreadsheet) can count how many times the sums 9, 11, 13, ... occur.

However, the following graphical method is easier to implement:

1 + 3 + 5 = 9 overlap to the right: 0 stones (0 + 0 + 0)	
1 + 3 + 7 = 11 overlap to the right: 1 stone (0 + 0 + 1)	
1 + 3 + 9 = 13 overlap to the right: 2 stones (0 + 0 + 2)	
1 + 5 + 7 = 13 overlap to the right: 2 stones (0 + 1 + 1)	

1 + 3 + 11 = 15 overlap to the right: 3 stones (0 + 0 + 3)	
1 + 5 + 9 = 15 overlap to the right: 3 stones (0 + 1 + 2)	
3 + 5 + 7 = 15 overlap to the right: 3 stones (1 + 1 + 1)	

The odd number of stones are arranged as L-shaped forms (see above), with the axis of symmetry lying in the diagonal of the square of side length 3.

The total number of stones can be read from the stones protruding to the right:

Total number = stones in the square + overlap above + overlap to the right = 9 + 2 · overlap to the right

If, for example, 5 stones protrude to the right, this means that the sum 9 + 2 · 5 = 19 is shown.

If you want to know in how many ways certain sums can be illustrated, you only have to consider in how many ways a certain protrusion is possible. The three summands are noted in ascending order, starting at 0:

sum	overlap	combinations
9	0	0 + 0 + 0
11	1	0 + 0 + 1
13	2	0 + 0 + 2, 0 + 1 + 1
15	3	0 + 0 + 3, 0 + 1 + 2, 1 + 1 + 1
17	4	0 + 0 + 4, 0 + 1 + 3, 0 + 2 + 2, 1 + 1 + 2
19	5	0 + 0 + 5, 0 + 1 + 4, 0 + 2 + 3, 1 + 1 + 3, 1 + 2 + 2
…	…	…

✱ A 4.3:

By systematic trial and error you can find

1 possibility for the sum 16π: $1\pi + 3\pi + 5\pi + 7\pi$

1 possibility for the sum 18π: $1\pi + 3\pi + 5\pi + 9\pi$

2 possibilities for the sum 20π: $1\pi + 3\pi + 5\pi + 11\pi$, $1\pi + 3\pi + 7\pi + 9\pi$

3 possibilities for the sum 22π: $1\pi + 3\pi + 5\pi + 13\pi$, $1\pi + 3\pi + 7\pi + 11\pi$, $1\pi + 5\pi + 7\pi + 9\pi$

5 possibilities for the sum 24π:
$1\pi + 3\pi + 5\pi + 15\pi$, $1\pi + 3\pi + 7\pi + 13\pi$, $1\pi + 3\pi + 9\pi + 11\pi$, $1\pi + 5\pi + 7\pi + 11\pi$, $3\pi + 5\pi + 7\pi + 9\pi$,

6 possibilities for the sum 26π:
$1\pi + 3\pi + 5\pi + 17\pi$, $1\pi + 3\pi + 7\pi + 15\pi$, $1\pi + 3\pi + 9\pi + 13\pi$, $1\pi + 5\pi + 7\pi + 13\pi$, $1\pi + 5\pi + 9\pi + 11\pi$, $3\pi + 5\pi + 7\pi + 11\pi$

9 possibilities for the sum 28π:
$1\pi + 3\pi + 5\pi + 19\pi$, $1\pi + 3\pi + 7\pi + 17\pi$, $1\pi + 3\pi + 9\pi + 15\pi$, $1\pi + 3\pi + 11\pi + 13\pi$, $1\pi + 5\pi + 7\pi + 15\pi$, $1\pi + 5\pi + 9\pi + 13\pi$, $1\pi + 7\pi + 9\pi + 11\pi$, $3\pi + 5\pi + 7\pi + 13\pi$, $3\pi + 5\pi + 9\pi + 11\pi$

11 possibilities for the sum 30π:
$1\pi + 3\pi + 5\pi + 21\pi$, $1\pi + 3\pi + 7\pi + 19\pi$, $1\pi + 3\pi + 9\pi + 17\pi$, $1\pi + 3\pi + 11\pi + 15\pi$, $1\pi + 5\pi + 7\pi + 17\pi$, $1\pi + 5\pi + 9\pi + 15\pi$, $1\pi + 5\pi + 11\pi + 13\pi$, $1\pi + 7\pi + 9\pi + 13\pi$, $3\pi + 5\pi + 7\pi + 15\pi$, $3\pi + 5\pi + 9\pi + 13\pi$, $3\pi + 7\pi + 9\pi + 11\pi$

etc.

You could also start that way:

There is only 1 possibility to form a sum with 4 (different) summands by using the 4 numbers 1π, 3π, 5π, 7π:

$1\pi + 3\pi + 5\pi + 7\pi = 16\pi$.

There are $\binom{5}{4} = 5$ possibilities to form a sum with 4 (different) summands by using the numbers 1π, 3π, 5π, 7π, 9π. Thus there are 4 additional combinations to form a sum:

$1\pi + 3\pi + 5\pi + 9\pi = 18\pi$, $1\pi + 3\pi + 7\pi + 9\pi = 20\pi$, $1\pi + 5\pi + 7\pi + 9\pi = 22\pi$, $3\pi + 5\pi + 7\pi + 9\pi = 24\pi$.

There are $\binom{6}{4} = 15$ possibilities to form a sum with 4 (different) summands by using the numbers 1π, 3π, 5π, 7π, 9π, 11π. Thus there are 10 *additional* combinations to form a sum:

These are all such sums which contain 11π as summand, i. e. all possibilities with three summands which contain 1π, 3π, 5π, 7π or 9π as summands, i. e. $\binom{5}{3} = 10$:

$1\pi + 3\pi + 5\pi + 11\pi = 20\pi$, $1\pi + 3\pi + 7\pi + 11\pi = 22\pi$,

$1\pi + 3\pi + 9\pi + 11\pi = 1\pi + 5\pi + 7\pi + 11\pi = 24\pi$, $1\pi + 5\pi + 9\pi + 11\pi = 3\pi + 5\pi + 7\pi + 11\pi = 26\pi$,

$1\pi + 7\pi + 9\pi + 11\pi = 3\pi + 5\pi + 9\pi + 11\pi = 28\pi$, $3\pi + 7\pi + 9\pi + 11\pi = 30\pi$, $5\pi + 7\pi + 9\pi + 11\pi = 32\pi$.

When you consider the 7 numbers 1π, 3π, 5π, 7π, 9π, 11π, 13π there are $\binom{7}{4} = 35$ ways to form a sum with 4 summands, i. e. 20 *additional* combinations. These are all such sums which contain 13π as summand, i. e. all possibilities with three summands which contain 1π, 3π, 5π, 7π, 9π, 11π as summands, i. e. $\binom{6}{3} = 20$:

$1\pi + 3\pi + 5\pi + 13\pi = 22\pi$, $1\pi + 3\pi + 7\pi + 13\pi = 24\pi$, $1\pi + 3\pi + 9\pi + 13\pi = 1\pi + 5\pi + 7\pi + 13\pi = 26\pi$,

$1\pi + 3\pi + 11\pi + 13\pi = 1\pi + 5\pi + 9\pi + 13\pi = 3\pi + 5\pi + 7\pi + 13\pi = 28\pi$,

$1\pi + 5\pi + 11\pi + 13\pi = 1\pi + 7\pi + 9\pi + 13\pi = 3\pi + 5\pi + 9\pi + 13\pi = 30\pi$,

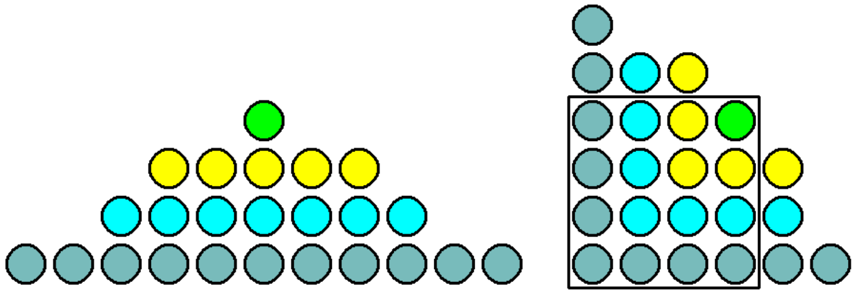
$1\pi + 7\pi + 11\pi + 13\pi = 3\pi + 5\pi + 11\pi + 13\pi = 3\pi + 7\pi + 9\pi + 13\pi = 32\pi$,

$1\pi + 9\pi + 11\pi + 13\pi = 3\pi + 7\pi + 11\pi + 13\pi = 5\pi + 7\pi + 9\pi + 13\pi = 34\pi$,

$3\pi + 9\pi + 11\pi + 13\pi = 5\pi + 7\pi + 11\pi + 13\pi = 36\pi$, $5\pi + 9\pi + 11\pi + 13\pi = 38\pi$, $7\pi + 9\pi + 11\pi + 13\pi = 40\pi$.

etc.

Another possibility is to determine the number of possibilities by means of dual numbers with a sum cross of 4 (see A 4.2). Or you can choose the graphical method described above, where you have to examine the overlap of a square with 16 stones ... (as shown in the figure, 1 + 5 + 7 + 11).



✱ A 4.4:

If both areas are equal in size, then the area of the inner circle is half the size of the total area. For the radii of the two circles, the following applies: $r_{outside} = \sqrt{2} \cdot r_{inside}$, because we have

green + light blue = $\pi \cdot r_{outside}^2 = 2 \cdot (\pi \cdot r_{inside}^2) = 2 \cdot$ light blue.

Since $\sqrt{2}$ is not a rational number, i. e. it cannot be represented as a fraction of two natural numbers, there are no suitable circular rings with the width 1.

The same applies to the second figure, where the equation $r_{outside} = \sqrt{3} \cdot r_{inside}$ should be fulfilled. In addition, the following should also apply: $r_{middle} = \sqrt{2} \cdot r_{inside}$

For the third figure, the radius of the outer circle is twice as large as that of the blue circle on the inside (the area of the red circle is half the total area), but the following should also apply: $r_{second\ circle\ inside} = \sqrt{2} \cdot r_{inside}$

and $r_{second\ circle\ inside} = \sqrt{3} \cdot r_{inside}$.

✱ A 4.5:

In the figures shown, the inner 5 circular rings are coloured light blue; together they have an area of $5^2 = 25$. The total figure contains 7 circular rings with an area of $7^2 = 49$, i.e. the green circular rings have an area of $7^2 - 5^2 = 24$.

It therefore applies: $\frac{7^2}{5^2} = \left(\frac{7}{5}\right)^2 \approx 2$. The fraction $\frac{7}{5}$ is therefore an approximation for $\sqrt{2}$. From the continued fraction expansion of $\sqrt{2}$ (see chapter 3) we know that better approximations can be obtained with with $\frac{a+2b}{a+b}$, i. e. $\frac{17}{12}$ and further $\frac{41}{29}$.

✱ A 4.6:

From the continued fraction expansion of $\sqrt{3}$ we have: $\frac{5}{3}, \frac{19}{11}, \frac{71}{41}$, i.e., the ratio of the areas are:

$25\pi : 9\pi$, $361\pi : 121\pi$, $5041\pi : 1681\pi$.

✱ A 4.7:

Dadurch, dass der Mittelpunkt der Kreise jeweils verschoben ist, nimmt die Möglichkeit ab, die Radien miteinander zu vergleichen.
The fact that the centres of the circles are shifted reduces our chances of comparing the radii.

✱ A 4.8:

(1) Shown are circular rings with r = 1 and r = 2, i.e. $A_{yellow} = 1\pi$ and $A_{green} = A_{light\ blue} = A_{blue} = (3\pi)/3 = 1\pi$

It could also be that multiples of the radii were used for the picture, for example r = 2 and r = 4: $A_{yellow} = (1 + 3) \cdot \pi = 4\pi$ and $A_{green} = A_{light\ blue} = A_{blue} = (5\pi + 7\pi)/3 = (12\pi)/3 = 4\pi$

The same applies to the following illustrations.

(2) $A_{pink} = 1\pi$; $A_{blue\text{-}violet} = (5\pi)/5 = 1\pi$

(3) $A_{red} = 1\pi$; $A_{pink} = (7\pi)/7 = 1\ \pi$

(4) $A_{yellow} = 1\pi + 3\pi = 4\pi$; $A_{light\ blue} = (7\pi + 9\pi)/4 = 4\pi$

(5) $A_{pink} = 1\pi$; $A_{blue/golden} = (3\pi + 5\pi)/8 = 1\pi$

(6) $A_{blue} = 3\pi + 5\pi = 8\pi$; $A_{blue\text{-}grey} = (7\pi + 9\pi + 11\pi + 13\pi)/5 = 8\pi$

(7) $A_{orange} = 3\pi$; $A_{brown} = (7\pi + 9\pi + 11\pi)/9 = 3\pi$

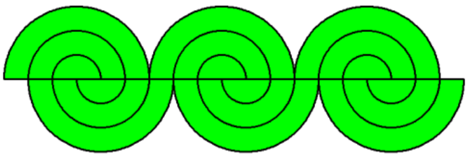
(8) $A_{olive} = 5\pi + 7\pi = 12\pi$; $A_{green} = (9\pi + 11\pi + 13\pi + 15\pi)/4 = 12\pi$

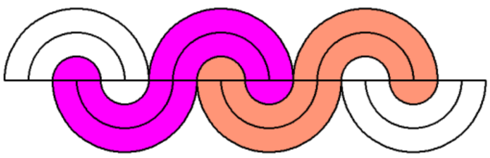
✱ A 4.9:

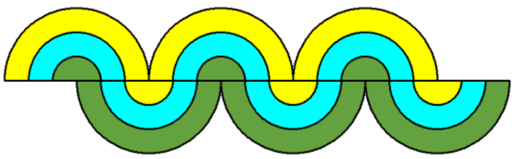
- Figure 1: The total area of 49π colored orange.
- Figure 2: The green colored area has the same size as the light blue colored area:
  $\frac{1}{2} \cdot (1\pi + 3\pi + 5\pi + 7\pi + 9\pi + 11\pi + 13\pi) = \frac{1}{2} \cdot 49\pi = 24{,}5 \cdot \pi$
- Figure 3: green = violet: $\frac{1}{2} \cdot (1\pi + 5\pi + 7\pi + 11\pi + 13\pi) = \frac{1}{2} \cdot 37\pi = 18{,}5 \cdot \pi$
  blue-green: $3\pi + 9\pi = 12\pi$
- Figure 4: yellow = orange: $\frac{1}{2} \cdot (1\pi + 7\pi + 9\pi) = \frac{1}{2} \cdot 17\pi = 8{,}5 \cdot \pi$
  red = pink: $\frac{1}{2} \cdot (3\pi + 5\pi + 11\pi + 13\pi) = \frac{1}{2} \cdot 32\pi = 16\pi$
- Figure 5: red = orange: $\frac{1}{2} \cdot (1\pi + 9\pi + 11\pi) = \frac{1}{2} \cdot 21\pi = 10{,}5 \cdot \pi$
  pink = yellow: $\frac{1}{2} \cdot (3\pi + 7\pi + 13\pi) = \frac{1}{2} \cdot 23\pi = 11{,}5 \cdot \pi$
  brown: 5π
- Figure 6: grey = blue-violet: $\frac{1}{2} \cdot (1\pi + 11\pi + 13\pi) = \frac{1}{2} \cdot 25\pi = 12{,}5 \cdot \pi$
  blue = purple: $\frac{1}{2} \cdot (3\pi + 9\pi) = \frac{1}{2} \cdot 12\pi = 6\pi$
  hellblau = rot : $\frac{1}{2} \cdot (5\pi + 7\pi) = \frac{1}{2} \cdot 12\pi = 6\pi$
- Figure 7: light blue = blue-violet: $\frac{1}{2} \cdot (1\pi + 13\pi) = \frac{1}{2} \cdot 14\pi = 7\pi$
  green = purple: $\frac{1}{2} \cdot (3\pi + 11\pi) = \frac{1}{2} \cdot 14\pi = 7\pi$
  yellow = red: $\frac{1}{2} \cdot (5\pi + 9\pi) = \frac{1}{2} \cdot 14\pi = 7\pi$
  orange: 7π
- Figure 8: brown = blue-violet: $\frac{1}{2} \cdot 1\pi = 0{,}5 \cdot \pi$
  pink = blue-grey: $\frac{1}{2} \cdot (3\pi + 13\pi) = \frac{1}{2} \cdot 16\pi = 8\pi$
  yellow = light blue: $\frac{1}{2} \cdot (5\pi + 11\pi) = \frac{1}{2} \cdot 16\pi = 8\pi$
  orange = green: $\frac{1}{2} \cdot (7\pi + 9\pi) = \frac{1}{2} \cdot 16\pi = 8\pi$
- Figure 9: blue = brown: $\frac{1}{2} \cdot 1\pi = 0{,}5 \cdot \pi$
  blue-violet = orange: $\frac{1}{2} \cdot 3\pi = 1{,}5 \cdot \pi$
  grey = red: $\frac{1}{2} \cdot (5\pi + 13\pi) = \frac{1}{2} \cdot 18\pi = 9\pi$
  green = purple: $\frac{1}{2} \cdot (7\pi + 11\pi) = \frac{1}{2} \cdot 18\pi = 9\pi$
  light blue: 9π

✱ A 4.10:

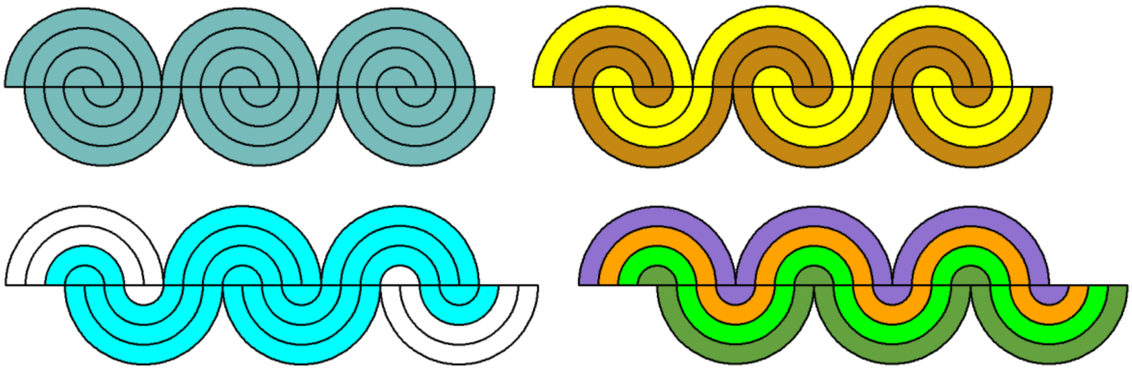
a)



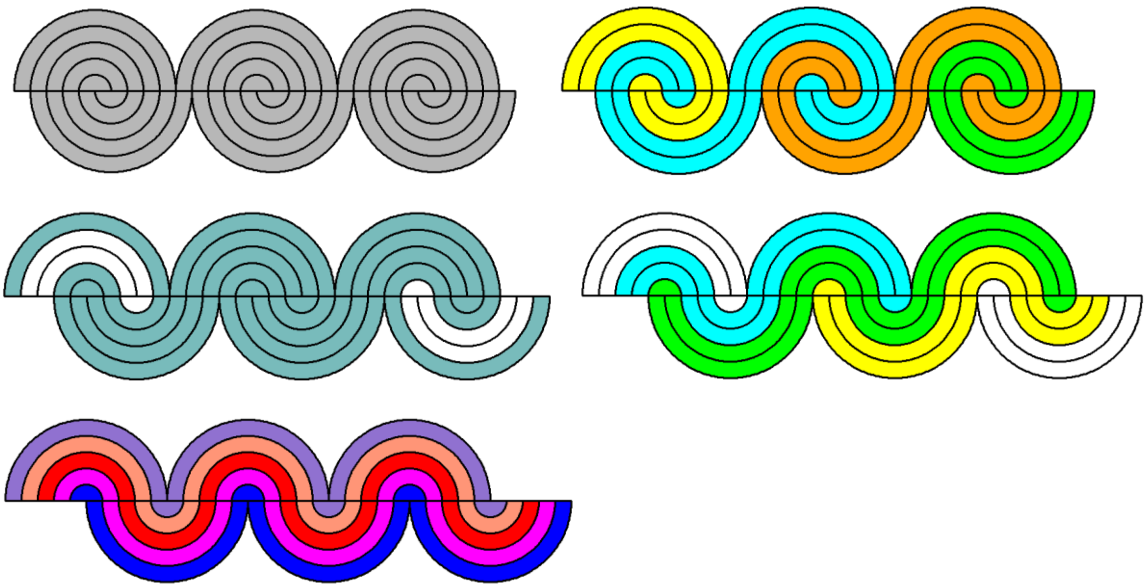




b)

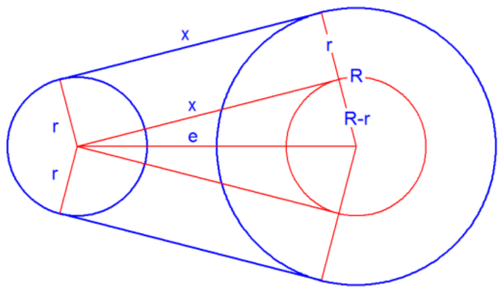


c)



✱ A 4.11: And what other patterns did you discover?

✱ A 4.12:

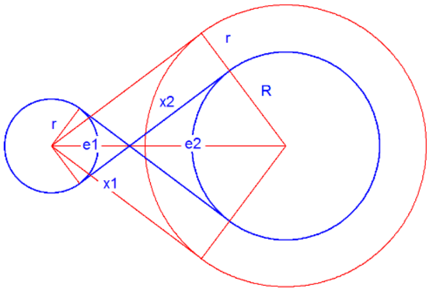


With the help of sin() = $(R – r)/e$ the angle can be determined from the given values of r, R and e. The total length u of the path results from the lengths of the arcs on the two circles and the two straight lines of length x. The length x can be calculated using the Pythagorean theorem or with: $\cos(\alpha) = x/e$:

$$u = 2x + b_r + b_R = 2e \cdot \cos(\alpha) + \frac{2\pi \cdot r \cdot (180° - 2\alpha)}{360°} + \frac{2\pi \cdot R \cdot (180° + 2\alpha)}{360°}$$

$$= 2e \cdot \cos(\alpha) + \pi \cdot r \cdot \left(1 - \frac{\alpha}{90°}\right) + \pi \cdot R \cdot \left(1 + \frac{\alpha}{90°}\right)$$

✱ A 4.13:



We indicate half of the angle of intersection (between the tangents) by α

For given radii r and R and the distance e = $e_1 + e_2$ between the centres, the following relationships results:

$\sin(\alpha) = \frac{R + r}{e}$; from this α can be determined. Further applies: $\tan(\alpha) = \frac{R + r}{x}$.

From this follows: $x = \frac{R + r}{\tan(\alpha)}$, so for the length u of the raceway

$$u = 2 \cdot \frac{R + r}{\tan(\alpha)} + \frac{2\pi \cdot r \cdot (180° + 2\alpha)}{360°} + \frac{2\pi \cdot R \cdot (180° + 2\alpha)}{360°} = 2 \cdot \frac{R + r}{\tan(\alpha)} + \pi \cdot (R + r) \cdot \left(1 + \frac{\alpha}{90°}\right)$$

The middle of several tracks is determined by the radii r and R and the distance e. Therefore the track which lies inside on the left side will have the radius r – d, and on the right side the radius is R + d, where d is the distance between the centre lines of the track. The distance e between the centres of the circles and the angle of intersection remain the same. Therefore the lengths of the two path are equal. (The same applies to the outer path on the left / inner path on the right).

If the sum R + r is replaced by e · sin(α) in the expression for u, we have

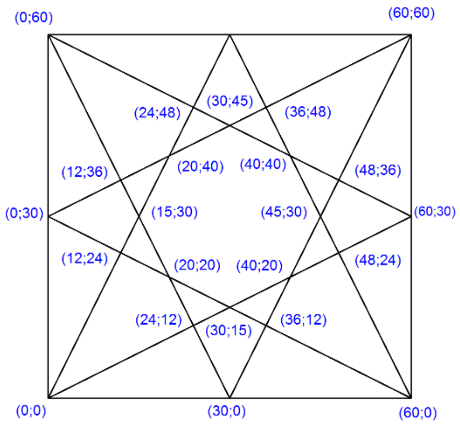
$$u = 2 \cdot \frac{e \cdot \sin(\alpha)}{\tan(\alpha)} + \pi \cdot e \cdot \sin(\alpha) \cdot \left(1 + \frac{\alpha}{90°}\right) = e \cdot \left[2 \cdot \cos(\alpha) + \pi \cdot \sin(\alpha) \cdot \left(1 + \frac{\alpha}{90°}\right)\right]$$

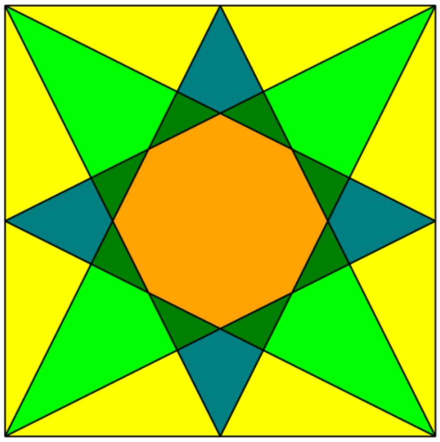
If one would set up such a track in a typical stadium with a 400 m track (r = R = 36.80 cm, e = 84.40 m), then α ≈ 60.7° and the running track would have a length of about 470 m.

Chapter 6

✱ A 6.1:

The integer coordinates of the intersection points result from the three figures with grid lines.





areas:

orange: 600 area units, gelb: 180 (each); light green: 240 (each);
blue-green 90 (each); dark green: 30 (each).

In the figure there are several rectangular triangles of different sizes:

- yellow colored triangles: hypotenuse: 30, other sides: $12 \cdot \sqrt{5}$ and $6 \cdot \sqrt{5}$

and two triangles, which are similar to these:

- vertices (0,0); (60,0) and (48,24):
  hypotenuse: 60, other sides: $24 \cdot \sqrt{5}$ and $12 \cdot \sqrt{5}$
- vertices (0,0), (60,0) und (60,30):
  hypotenuse: $30 \cdot \sqrt{5}$, other sides: 60 and 30

Further the rectangular triangle with the vertices (0,0) ; (24,48) and (60,30):

- hypotenuse: $30 \cdot \sqrt{5}$, other sides: $18 \cdot \sqrt{5}$ and $24 \cdot \sqrt{5}$ (ratio: 3:4:5)

and the triangle which is similar with the vertices (0,0) ; (30,15) and (12,24)

- hypotenuse: $15 \cdot \sqrt{5}$, other sides: $12 \cdot \sqrt{5}$ and $18 \cdot \sqrt{5}$

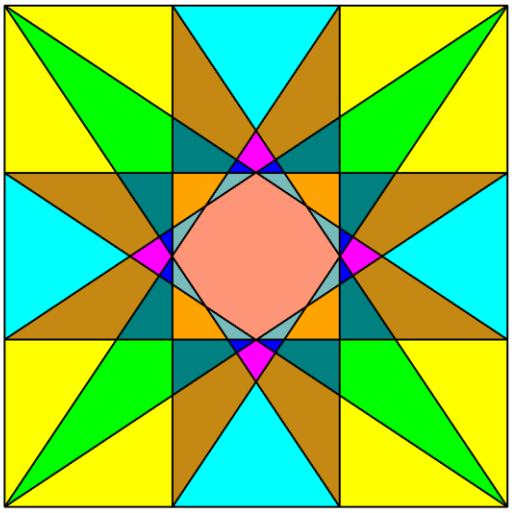
and the dark green colored triangles

- hypotenuse: $5 \cdot \sqrt{5}$, other sides: $3 \cdot \sqrt{5}$ and $4 \cdot \sqrt{5}$

and the blue-green colored quadrilateral together with the dark-green colored triangle form a rectangular triangle which is similar with

- hypotenuse: $10 \cdot \sqrt{5}$, other sides: $6 \cdot \sqrt{5}$ and $8 \cdot \sqrt{5}$

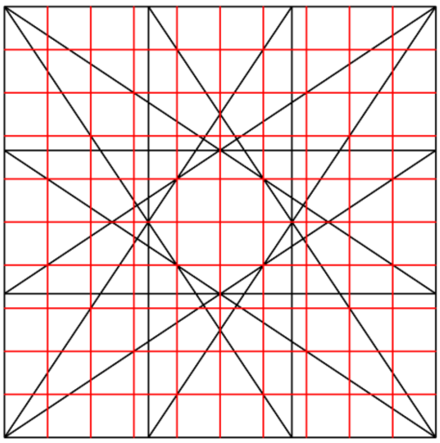
✱ A 6.2:

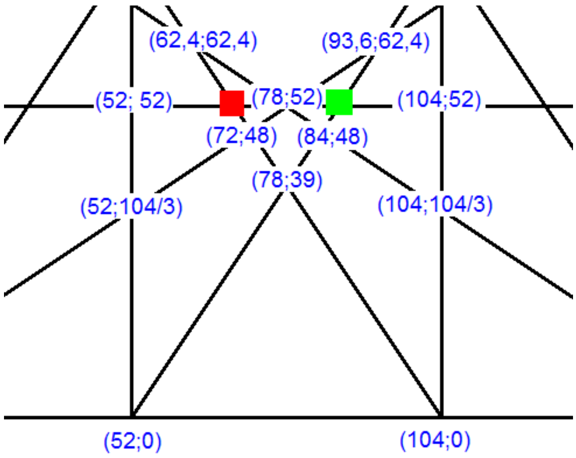


The grid lines divide the basic sides of the square into 12 or 13 equally sized segments. Therefore it makes sense to choose a length 12 · 13 = 156 for the square sides.

However, not all intersection points have integer coordinates. For example, you can find further intersection points on grid lines if you divide the basic sides of the square into 10 equally sized segments.

The red marked intersection point in the figure on the right has the coordinates (208/3 , 52), the green marked point has the coordinates (260/3 , 52).





Chapter 7

✱ A 7.1:

- **Check of the final digits:** 58^2 has the same final digits as $8^2 = 64$.
- **Method 1 (stepwise calculation)**

Starting with: $55^2 = 50 \cdot 60 + 5^2 = 3025$

$58^2 = 55^2 + (55 + 56) + (56 + 57) + (57 + 58) = 55^2 + 6 \cdot 56{,}5 = 3025 + 339 = 33\underline{64}$

Hint: 56,5 is the mean value of the six summands.

Starting with: $60^2 = 3600$

$58^2 = 60^2 - (60 + 59) - (59 + 58) = 60^2 - 4 \cdot 59 = 3600 - 236 = 33\underline{64}$

Hint: 59 is the mean value of the four summands.

- **Method 2 (calculation using equidistant numbers – a number of tens and a symmetrical partner)**

$58^2 = (60 - 2) \cdot (56 + 2) = 60 \cdot 56 + 2^2 = 3360 + 4 = 33\underline{64}$

$58^2 = (50 + 8) \cdot (66 - 8) + 8^2 = 50 \cdot 66 + 8^2 = 3300 + 64 = 33\underline{64}$

- **Method 3 (applying the 1st binomial formula)**

$58^2 = (50 + 8)^2 = 50^2 + 2 \cdot 50 \cdot 8 + 8^2 = 50 \cdot (50 + 2 \cdot 8) + 8^2 = 50 \cdot 66 + 8^2 = 3300 + 64 = 33\underline{64}$

$58^2 = (60 - 2)^2 = 60^2 - 2 \cdot 60 \;\square\; 2 + 2^2 = 60 \cdot (60 - 2 \cdot 2) + 2^2 = 60 \cdot 56 + 4 = 3360 + 4 = 3364$

- **Check of the final digits:** 84^2 has the same final digits as$16^2 = 2\underline{56}$.
- **Method 1 (stepwise calculation)**

Starting with: $85^2 = 80 \cdot 90 + 5^2 = 7225$

$84^2 = 85^2 - (85 + 84) = 85^2 - 2 \cdot 84{,}5 = 7225 - 169 = 70\underline{56}$

Starting with: $80^2 = 6400$

$84^2 = 80^2 + (80 + 81) + (81 + 82) + (82 + 83) + (83 + 84) = 80^2 + 8 \cdot 82 = 6400 - 656 = 70\underline{56}$

- **Method 2 (calculation using equidistant numbers – a number of tens and a symmetrical partner)**

$84^2 = (80 + 4) \cdot (88 - 4) = 80 \cdot 88 + 4^2 = 7040 + 16 = 70\underline{56}$

$84^2 = (90 - 6) \cdot (78 + 6) + 6^2 = 90 \cdot 78 + 6^2 = 7020 + 36 = 70\underline{56}$

Method 3 (applying the 1st binomial formula)

$84^2 = (80 + 4)^2 = 80^2 + 2 \cdot 80 \cdot 4 + 4^2 = 80 \cdot (80 + 2 \cdot 4) + 4^2 = 80 \cdot 88 + 4^2 = 7040 + 16 = 70\underline{56}$

$84^2 = (90 - 6)^2 = 90^2 - 2 \cdot 90 \cdot 6 + 6^2 = 90 \cdot (90 - 2 \cdot 6) + 6^2 = 90 \cdot 78 + 6^2 = 7020 + 36 = 70\underline{56}$

- **Check of the final digits:** 73^2 has the same final digits as $23^2 = 5\underline{29}$.
- **Method 1 (stepwise calculation)**

Starting with: $75^2 = 70 \cdot 80 + 5^2 = 5625$

$73^2 = 75^2 - (75 + 74) - (74 + 73) = 75^2 - 4 \cdot 74 = 5625 - 296 = 53\underline{29}$

Starting with: $70^2 = 4900$

$73^2 = 70^2 + (70 + 71) + (71 + 72) + (72 + 73) = 70^2 + 6 \cdot 71{,}5 = 4900 + 429 = 53\underline{29}$

- **Method 2 (calculation using equidistant numbers – a number of tens and a symmetrical partner)**

$73^2 = (70 + 3) \cdot (76 - 3) = 70 \cdot 76 + 3^2 = 5320 + 9 = 53\underline{29}$

$73^2 = (80 - 7) \cdot (66 + 7) + 7^2 = 80 \cdot 66 + 7^2 = 5280 + 49 = 53\underline{29}$

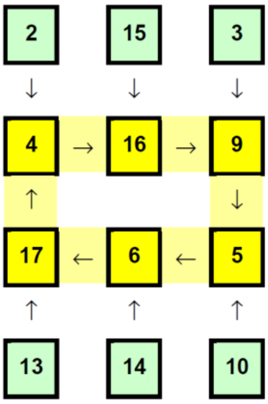
Method 3 (applying the 1st binomial formula)

$73^2 = (70 + 3)^2 = 70^2 + 2 \cdot 70 \cdot 3 + 3^2 = 70 \cdot (70 + 2 \cdot 3) + 3^2 = 70 \cdot 76 + 3^2 = 5320 + 9 = 53\underline{29}$

$73^2 = (80 - 7)^2 = 80^2 - 2 \cdot 80 \cdot 7 + 7^2 = 80 \cdot (80 - 2 \cdot 7) + 7^2 = 80 \cdot 66 + 7^2 = 5280 + 49 = 53\underline{29}$

✱ A 7.2:

(1) mod 19



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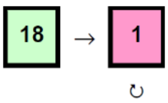
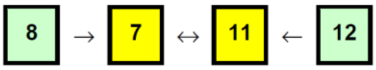
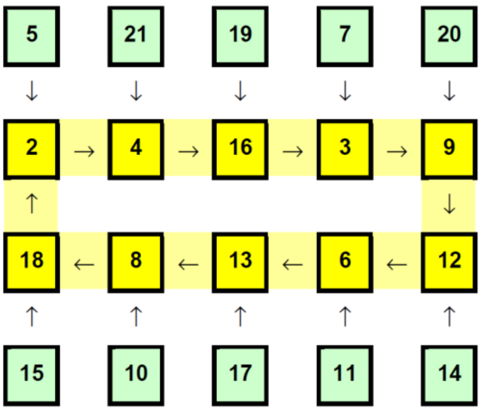




(2) mod 23



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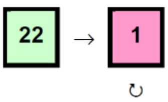
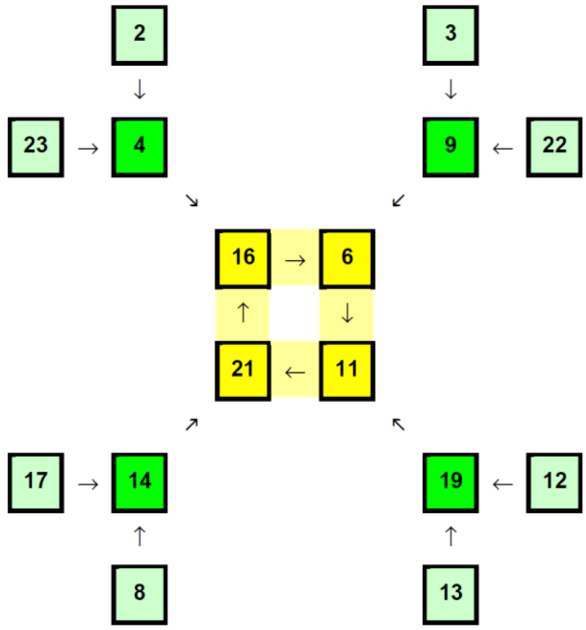


(3) mod 25



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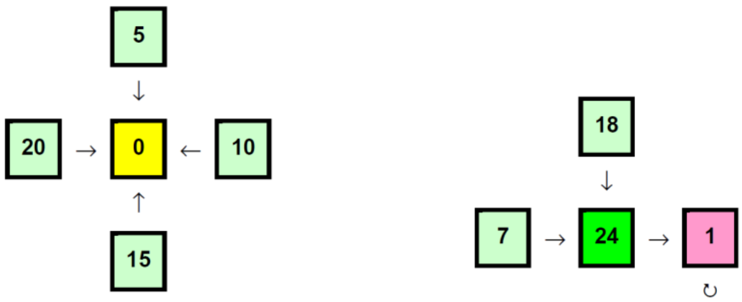
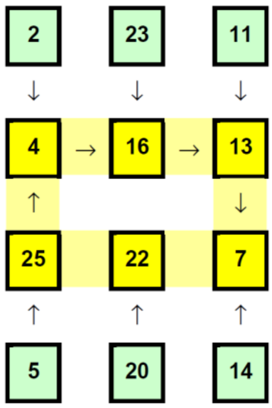


(4) mod 27



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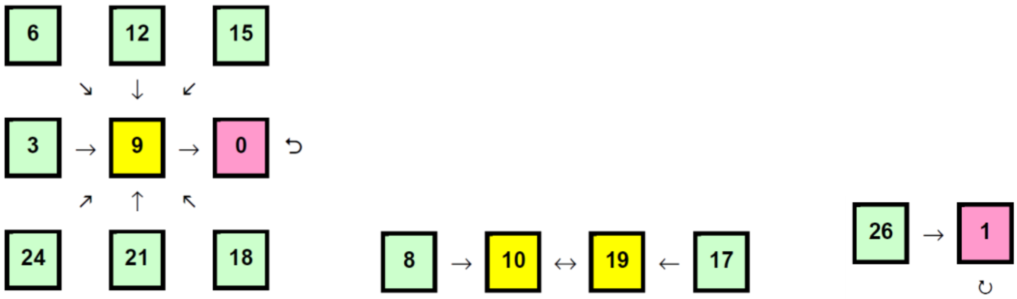
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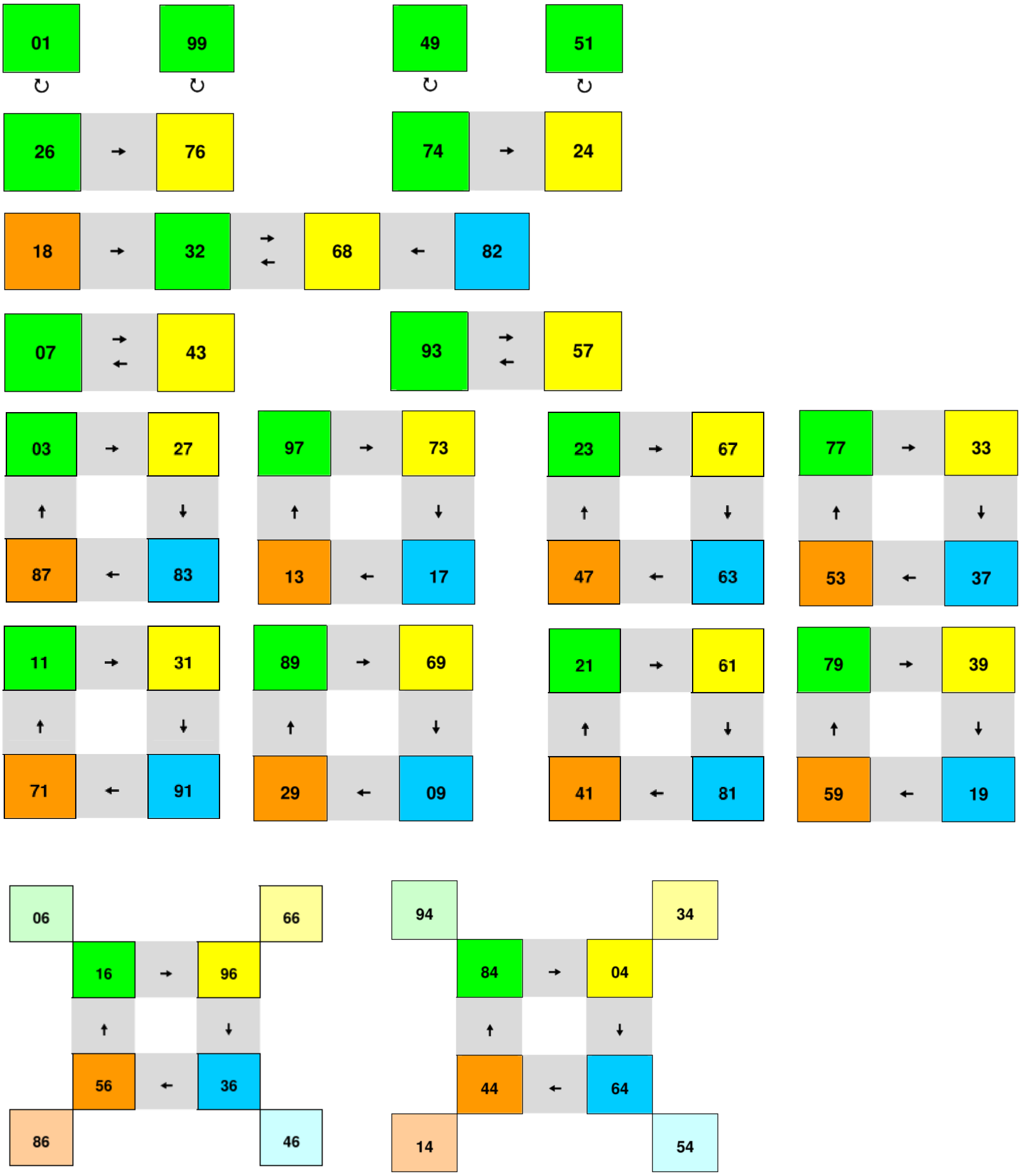
✱ A 7.3: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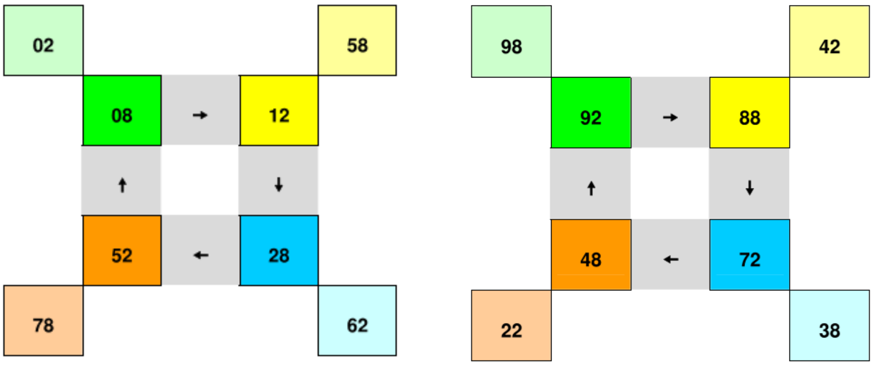
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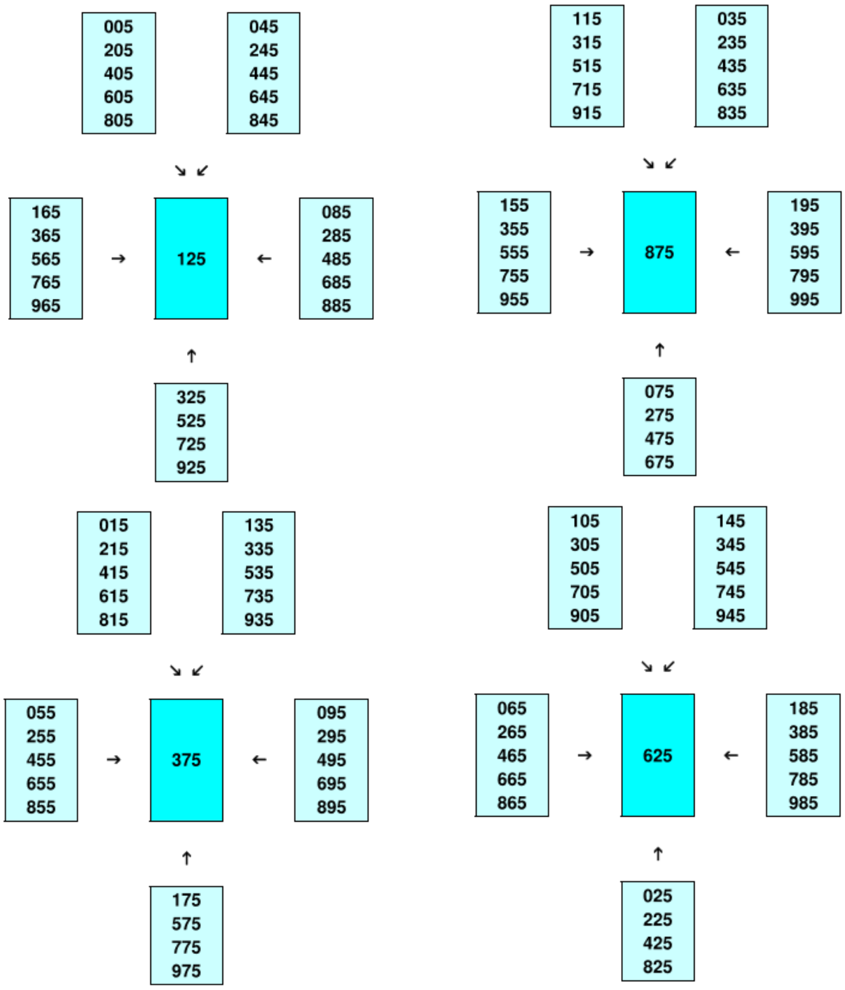
✱ A 7.4

(1)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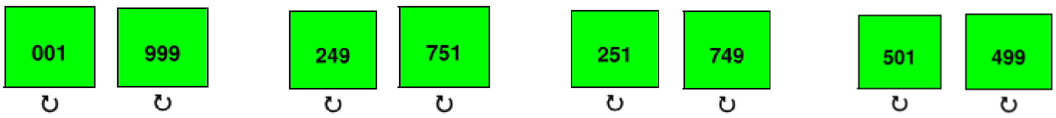
(2)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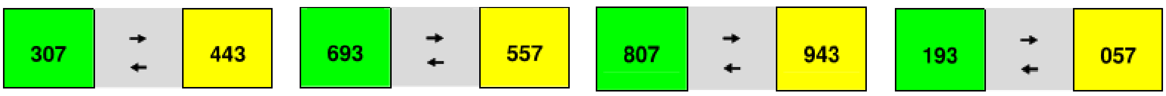
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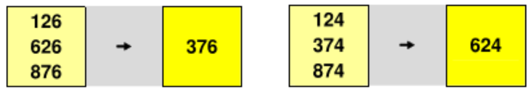
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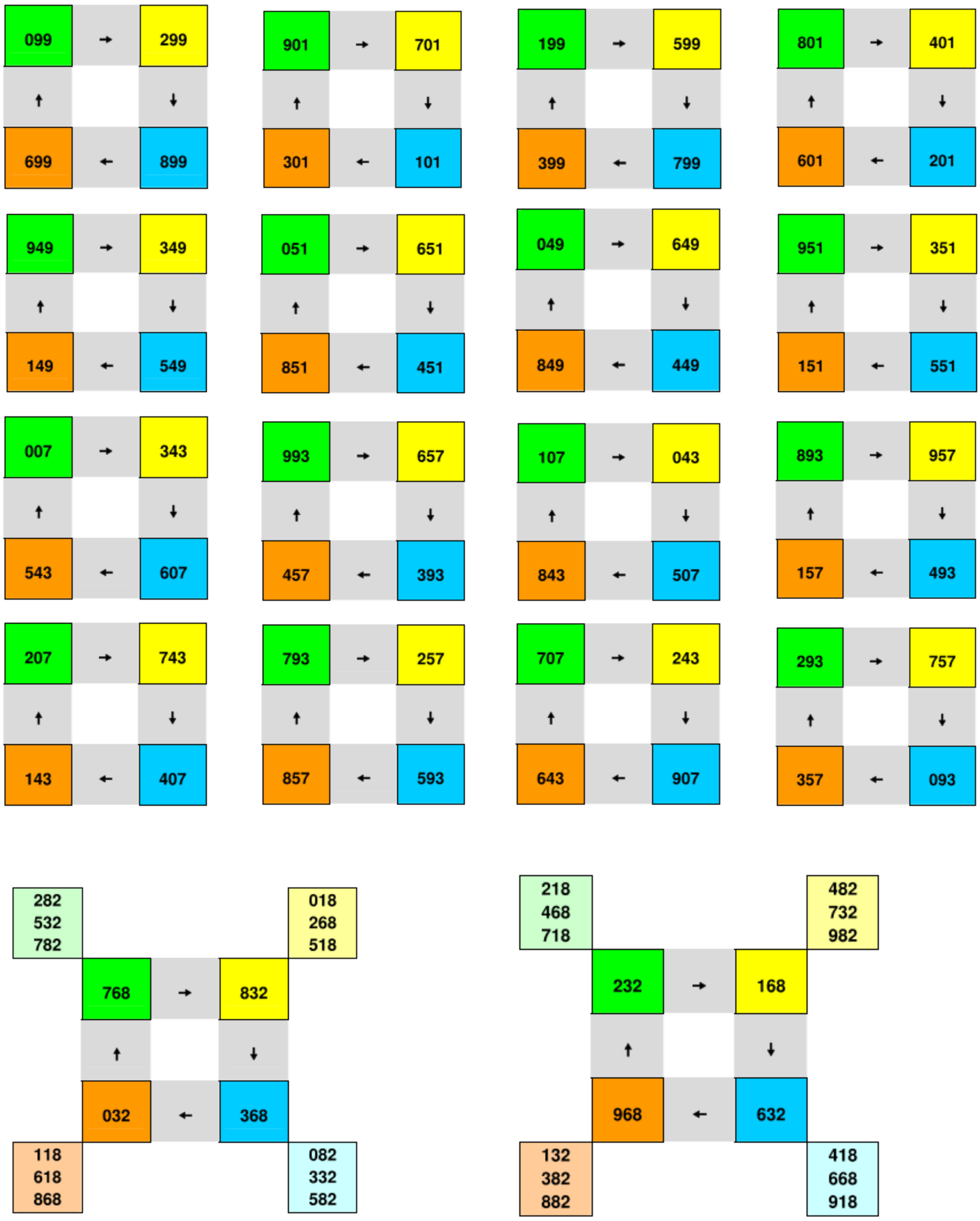
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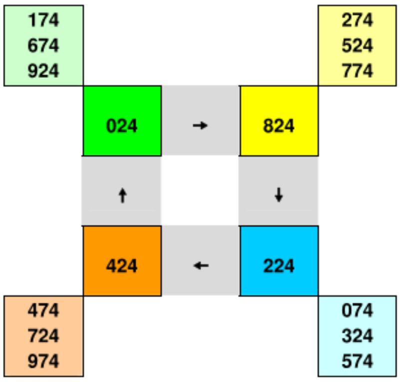
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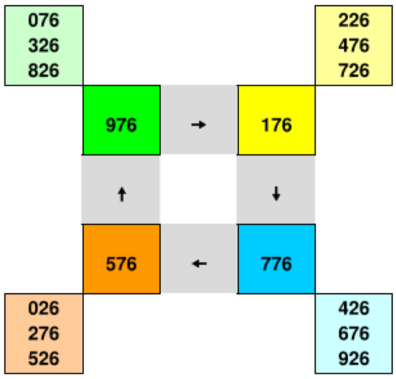
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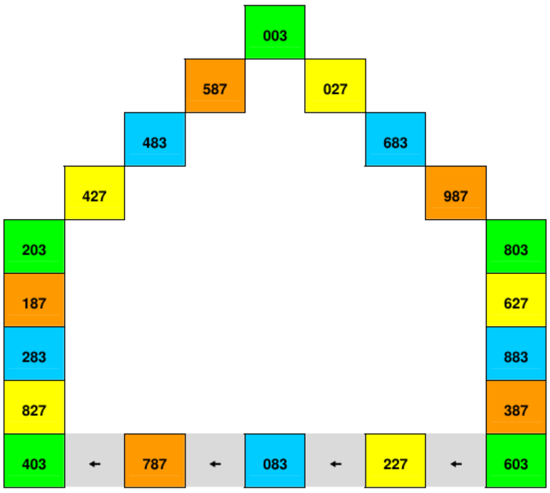
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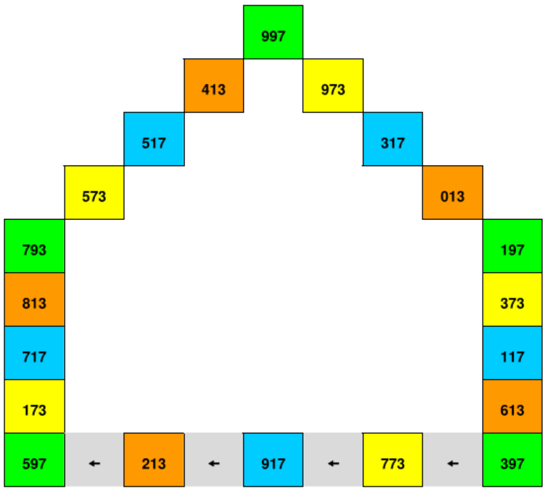
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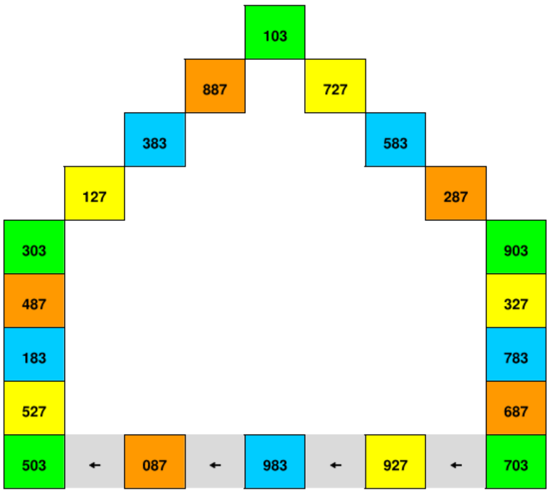
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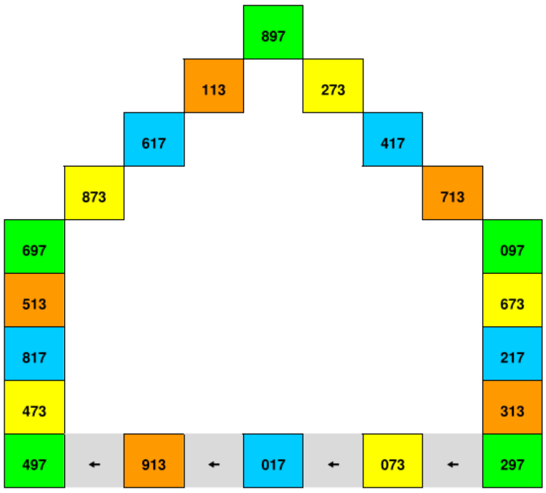
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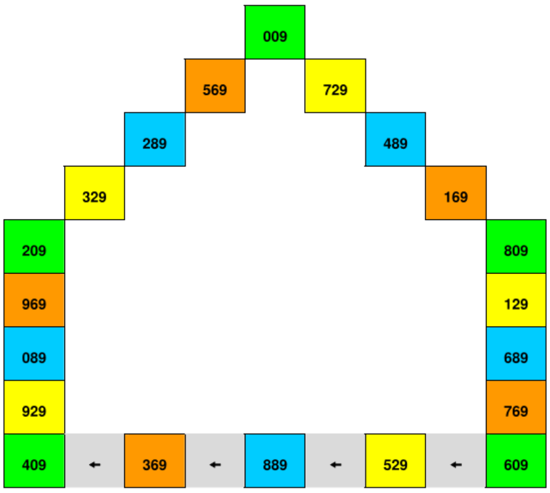
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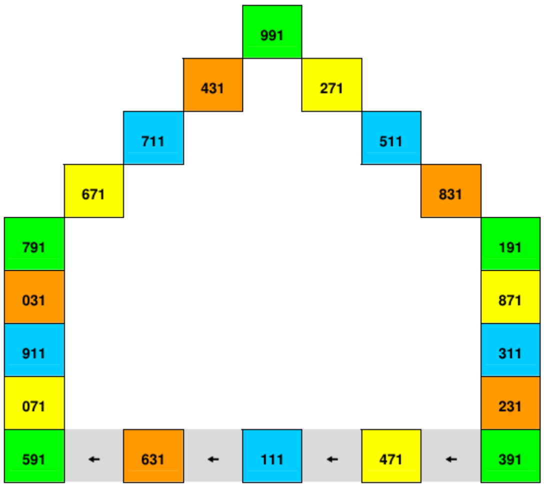
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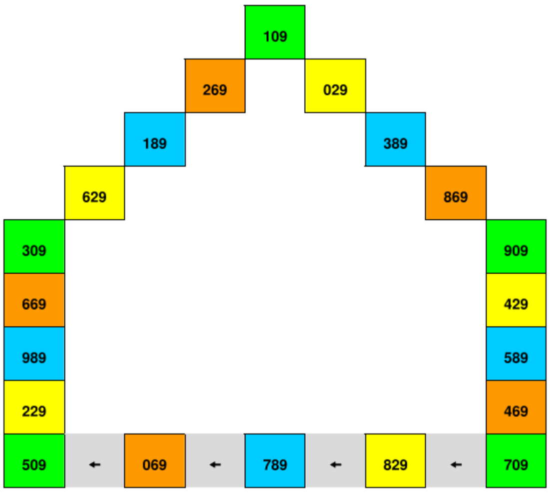
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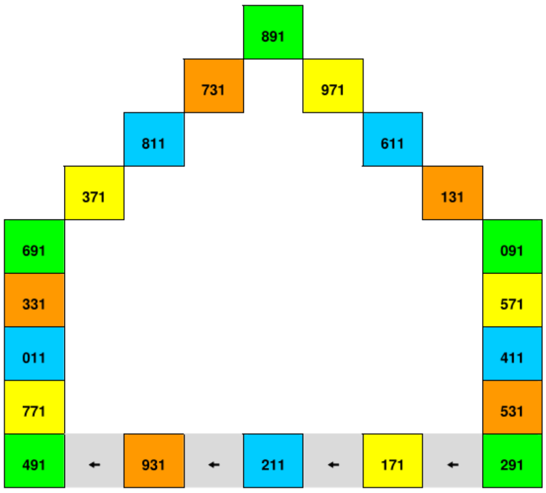
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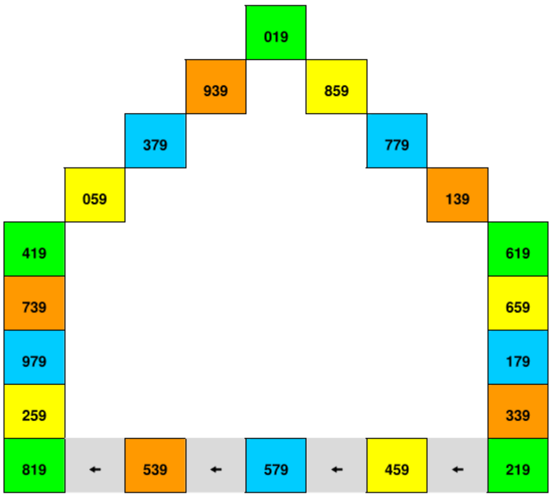
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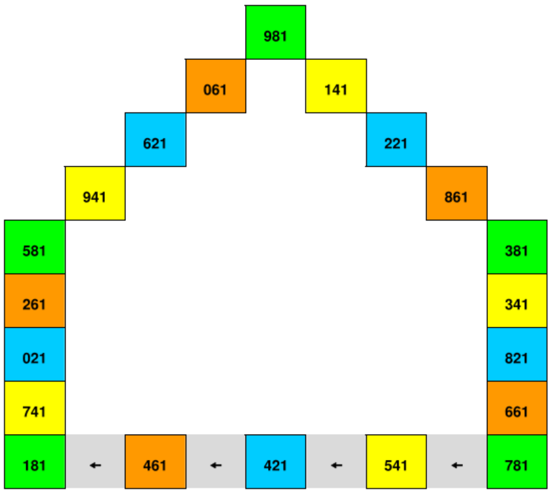
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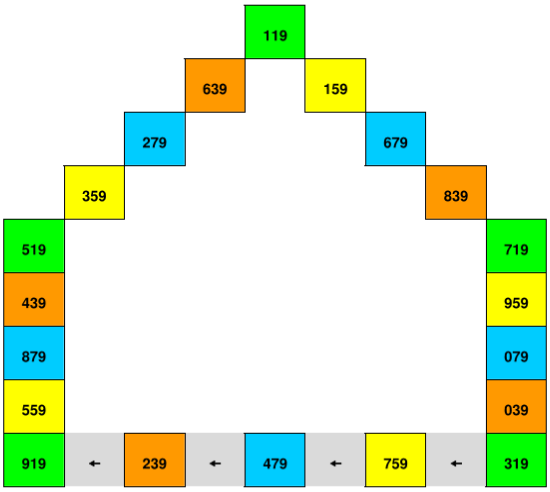
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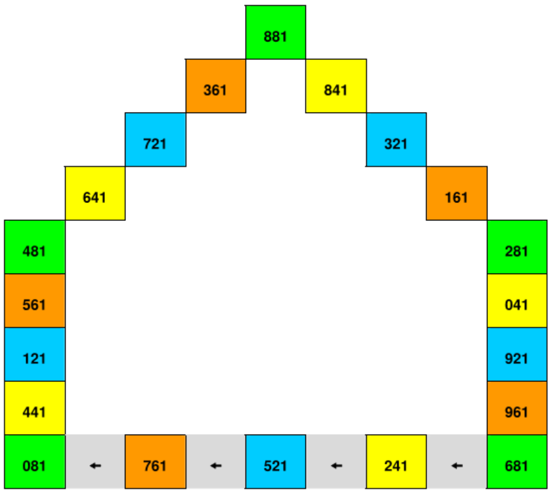
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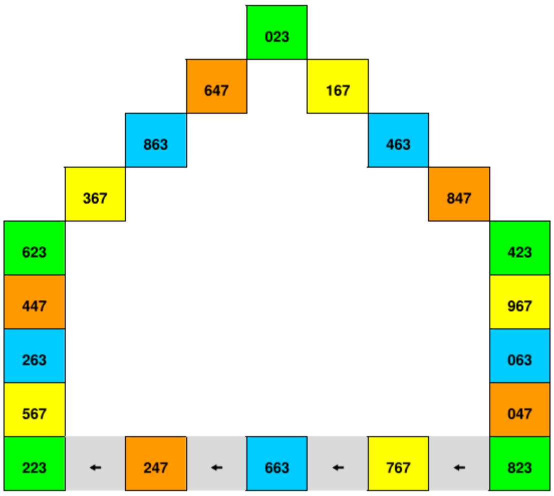
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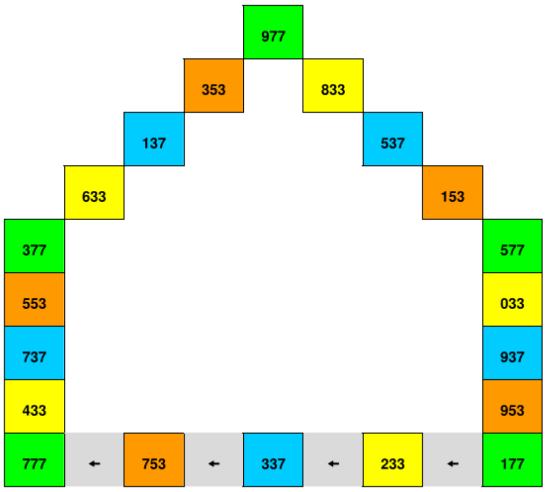
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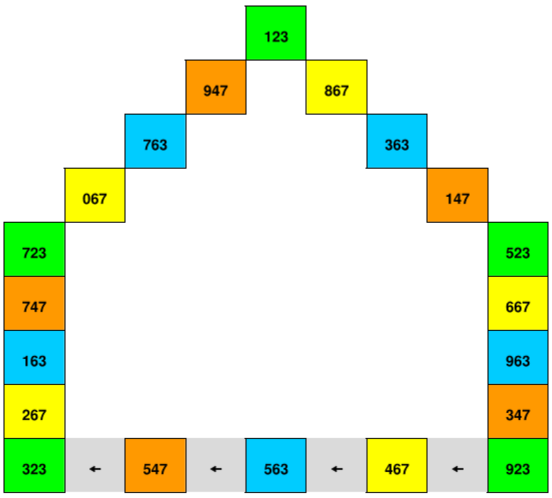
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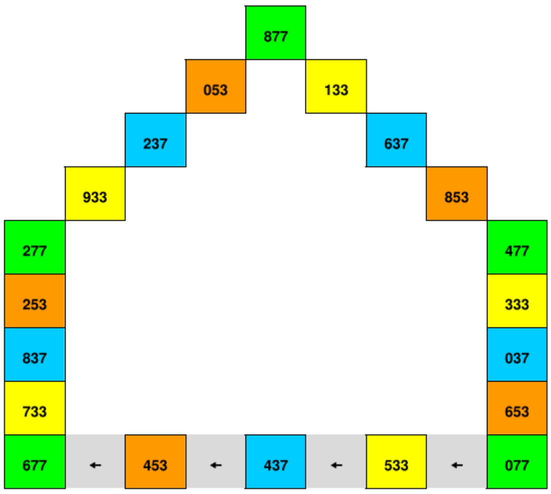
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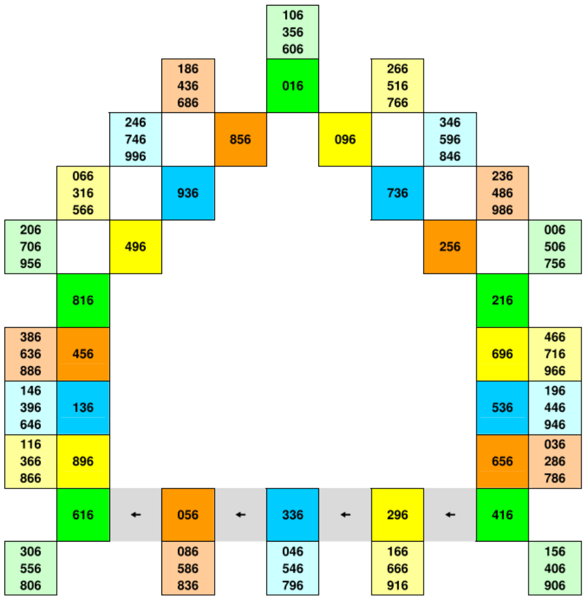
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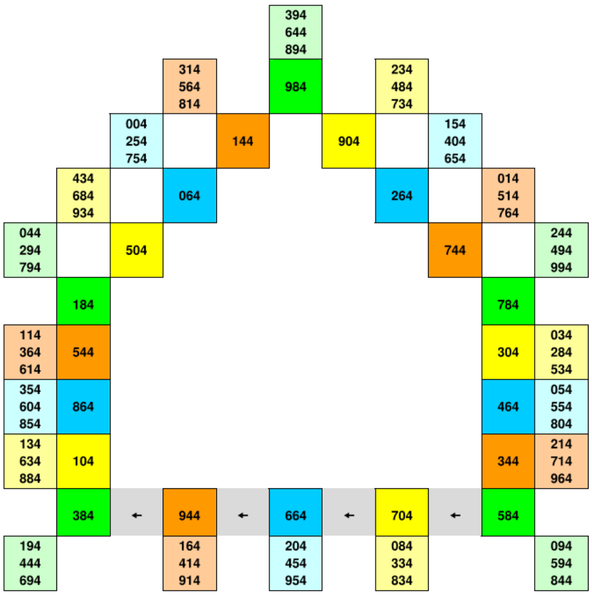
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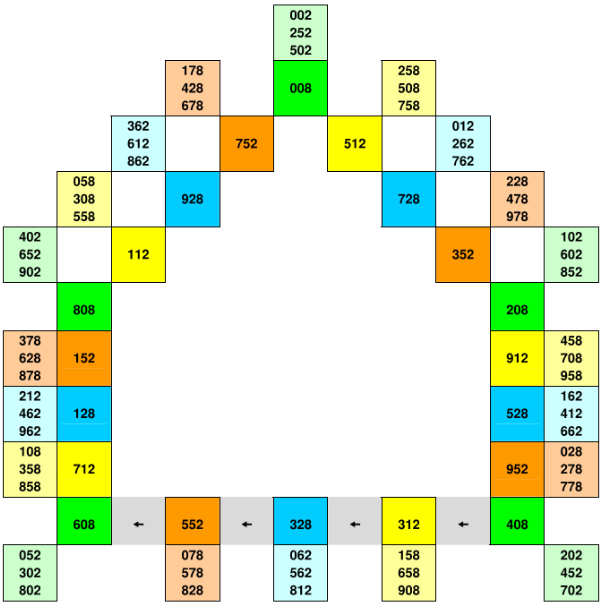
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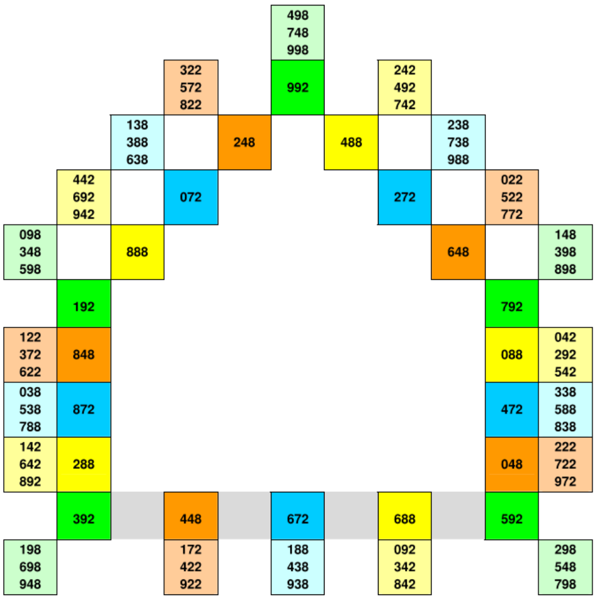
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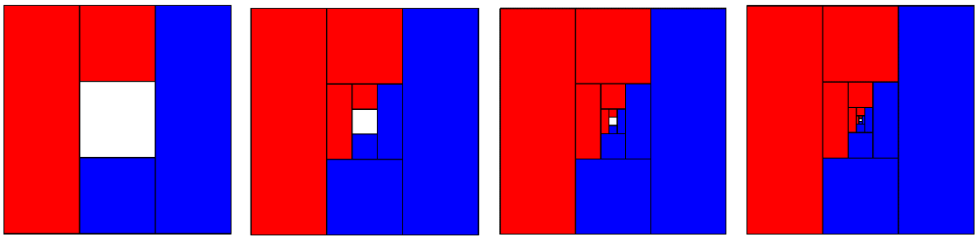
Chapter 8

✱ A 8.1:



✱ A 8.2:

When the white rectangle is divided by vertical lines into three parts, the red and blue colored areas "grow" together continuously.

✱ A 8.3:

Fig. on the left: The initial square was halved by a diagonal and one half was colored green, then the uncolored isosceles right-angled triangle was halved by the axis of symmetry and one half was colored yellow. In the following steps the uncolored triangle was halved and one of the triangles was colored alternately yellow and green. So we have

$$A_{green} = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots \quad \text{and} \quad A_{yellow} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots \right) = \frac{1}{2} \cdot A_{green}$$

Abb. rechts: Wie bei der Abb. in der Mitte wird zunächst das Ausgangsquadrat durch die beiden Diagonalen in vier gleichschenklig-rechtwinklige Dreiecke unterteilt. Die Färbung erfolgt aber in beiden Hälften jeweils entgegengesetzt. Daher ist die Quadratfläche jeweils zur Hälfte grün bzw. gelb gefärbt.

From this it follows that two thirds of the square is coloured green, one third yellow.

Fig. in the centre: In the first step the initial square is divided by the two diagonals into four isosceles right-angled triangles, two of which are colored green. The other two triangles are each halved by the axes of symmetry, one half of which is coloured yellow. From this it follows, as in the 1st figure, that two thirds of the square is colored green, one third yellow.

Fig. on the right: As in the middle figure, the initial square is first divided by the two diagonals into four isosceles right-angled triangles. However, the coloring is opposite in both halves. Therefore half of the square is colored green and half is colored yellow.

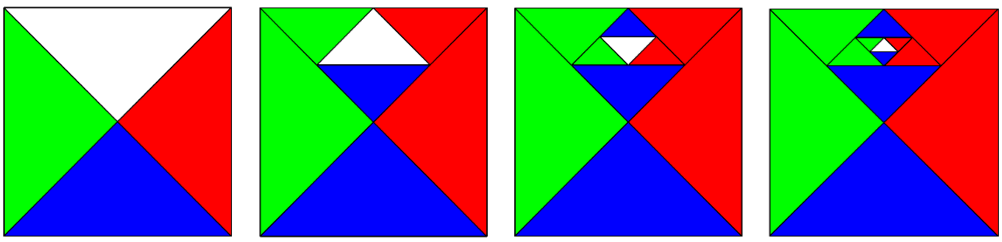
✱ A 8.4:

When we indicate the side length with a then we following applies for the area:

$$A = \frac{1}{2} \cdot a \cdot h = \frac{1}{2} \cdot a \cdot \frac{a}{2} \cdot \sqrt{3} = \frac{a^2}{4} \cdot \sqrt{3}\,.$$

From A = 1 it follows $a^2 = \frac{4}{\sqrt{3}}$, also $a = \frac{2}{\sqrt{\sqrt{3}}} = \frac{2}{\sqrt[4]{3}} \approx 1.52$

zu A 8.5:



✱ A 8.6:

Since the outer triangle is 4 times as large as the inner triangle, the side length of the inner triangle is $\frac{1}{\sqrt{4}}$ - times as large, i.e. ½ times as large as the outer triangle. So you draw the three outer symmetrical trapezoids with the parallel sides of length a and ½ · a and the base angle of 30°.

The oblique sides of the outer trapezoids are half as long as the distance of a vertex from the centre of the triangle (this is 2/3 times the altitude of the equilateral triangle), i.e. for the outer trapezoids this is $\frac{2}{3} \cdot \frac{a}{2} \cdot \sqrt{3} = \frac{a}{3} \cdot \sqrt{3} = \frac{a}{\sqrt{3}}$, and for the next trapezoids this is half the length of the next outer trapezoid.

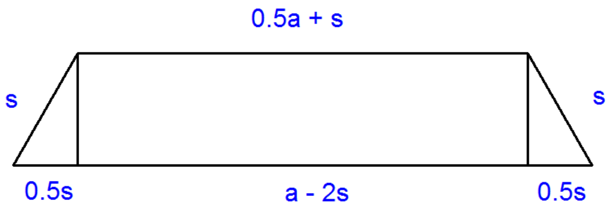
✱ A 8.7:

An equilateral triangle is divided into four equally sized areas in such a way that three congruent trapezoids are created on the outside and an equilateral triangle on the inside. As the outer triangle should be 4 times as large as the inner triangle, the side length of the inner triangle is $\frac{1}{\sqrt{4}}$ -times as large, i.e. ½ times as large as the outer triangle.

When we indicate the side length of the initial triangle with a, then in the first step the trapezoids have basic sides with the length a – s (bottom) and ½ · a + s (top) and two legs with the length s. Since the legs form an angle of 60° with the lower base, the following applies:

½ · a + s = a – 2s, i.e. s = 0.2 · a.

The lower base of the trapezoid therefore has the length 0.8 · a, the upper 0.7 · a, the legs 0.2 · a.



✱ A 8.8:

The first picture shows a square which is divided into five equally sized areas. If the side length of the square is $a_1 = 1$, the area of the square is 1.

The square in the middle of the figure then has a side length of $a_2 = \sqrt{\frac{1}{5}} \approx 0.4472$.

For the four rectangles with the side lengths b_2 and c_2 the following applies:

$b_2 + c_2 = 1$ and $b_2 \cdot c_2 = 1/5$, i. e. $b_2 = \frac{1}{5c_2}$ and therefore $b_2 + c_2 = \frac{1}{5c_2} + c_2 = \frac{1+5c_2^2}{5c_2} = 1$

This leads to the quadratic equation: $5c_2^2 - 5c_2 + 1 = 0$.

This has two positive solutions, namely $c_2 \approx 0.2764$ and $c_2 \approx 0.7236$..

Because $b_2 + c_2 = 1$, the following applies: $b_2 \approx 0.7236$ and $b_2 \approx 0.2764$

For the four rectangles enclosed in the square with the side length $a_2 = \sqrt{\frac{1}{5}} \approx 0.4472$ and the square in the middle, the following applies: $a_3 = \left(\sqrt{\frac{1}{5}}\right)^2 = \frac{1}{5}$ and further:

$c_3 \approx 0.2764 \cdot 0.4472 \approx 0.1236$ and $b_3 \approx 0.7236 \cdot 0.4472 \approx 0.3236$

We get the side length of the next figure by multiplying with 0.4472.

etc.

✱ A 8.9:

The lower base of the initial square is divided with the ratio 2 : 1 : 2. The left and right rectangles are each halved in the middle and the resulting four rectangles of equal size are colored in four colors.

The uncolored rectangle in the center is vertically divided with the ratio 2 : 1 : 2. The upper and lower rectangles are each divided lengthwise in the middle and the resulting four equal-sized rectangles are colored in four colors (as their neighbouring rectangles).

These two steps are repeated for the remaining uncoloured square in the middle.

✱ A 8.10:

Since the outer square should be 5 times as large as the inner square, the side length of the inner square is $\frac{1}{\sqrt{5}}$ - times as large as the outer square. The trapezoids therefore have basic sides of lengths a and (approx.) 0.447 · a. The altitude h of the trapezoids must fulfil the condition h + 0.447 · a + h = a, i.e. h ≈ 0.276 · a.

It follows that the legs, which form an angle of 45° and 135° with the two base sides, have a length of s = $\sqrt{2} \cdot h \approx$ 0.391 · a.

✱ A 8.11:

Since the outer square (with side length a) should be 5 times as large as the inner square, the side length of the inner square is $\frac{1}{\sqrt{5}}$ - times as large as the outer square, i.e. about 0.447 · a. This inner square has diagonals which are $\sqrt{2}$ -times as large as the side of this square, so they have a length of about 0.632 · a.

A partition of the initial square can therefore be performed as follows: The square is first divided into four equal squares by the two center lines. From the center of the figure, you measure half of 0.632 · a, i.e. a distance of 0.316 · a.

The four pentagons located in the vertices of the initial square thus have two sides of length 0.5 · a, which are perpendicular to each other, two sides of length 0.5 · a – 0.316 · a = 0.184 · a and the base side of the inner square with a length of 0.447 · a.

Appendix to ch. 8.5: Calculation of the side length of the regular pentagon (2nd method)

Since the outer pentagon should be 6 times as large as the inner pentagon, the side length of the inner pentagon is $\frac{1}{\sqrt{6}}$ - times as large as the outer pentagon.

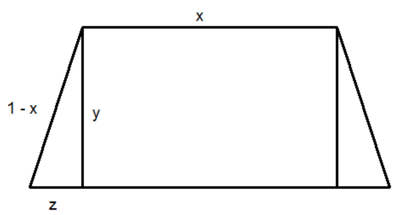
The area of a regular 5-sided polygon with side length a_1 = 1 can be calculated as follows:

$$A_1 = 5 \cdot \frac{1}{2} \cdot a_1 \cdot h_1 = \frac{5}{2} \cdot a_1 \cdot \frac{a_1}{2 \cdot \tan(36°)} = \frac{5a_1^2}{4 \cdot \tan(36°)}.$$

As a_1 = 1, this means: A_1 ≈ 1,7205.

Thus the inner pentagon has the area $A_2 = \frac{5a_1^2}{24 \cdot \tan(36°)}$.

This is also the area of the five symmetrical trapezoids. These trapezoids consist of a rectangle with the side lengths x (=upper side of the trapezoid) and y (= altitude of the trapezoid) and two right-angled triangles with the sides y and z; the hypotenuse of the right-angled triangle forms together with the upper side of the trapezoid a base side of the initial pentagon.



Therefore the area of a trapezoid is

$$A_2 = x \cdot y + 2 \cdot \frac{1}{2} \cdot z \cdot y = (x+z) \cdot y$$

Since y = (1 – x) · sin(72°) and z = (1 – x) · cos(72°) we have further

$$A_2 = [x + (1-x) \cdot \cos(72°)] \cdot (1-x) \cdot \sin(72°) = [x + \cos(72°) - x \cdot \cos(72°)] \cdot (1-x) \cdot \sin(72°)$$

The following equation must be solved by CAS

$$[x + \cos(72°) - x \cdot \cos(72°)] \cdot (1-x) \cdot \sin(72°) = \frac{5}{24 \cdot \tan(36°)}$$

Thus we get: x ≈ 0.5718 and from this y ≈ 0.4072 and z ≈ 0.1323.

We get the side lengths of the next figure by multiplication with $\sqrt{\frac{1}{6}} \approx 0.4082$.

✱ A 8.12:

Upper pictures: Since the radius is halved, the remaining yellow colored area covers a quarter of the total area of the octagon, i.e., for the light blue colored area of step 1 we have $A_1 = \frac{3}{4} \cdot A_{Octagon}$ and further for the green octagon, which covers three quarters of the remaining yellow-colored area:

$A_2 = \frac{3}{4} \cdot \frac{1}{4} \cdot A_{Octagon} = \frac{3}{16} \cdot A_{Octagon}$. Before we start step 3, one quarter of one quarter of the octagon is still yellow. In step 3, three quarters of it is then colored dark green: $A_3 = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot A_{Octagon} = \frac{3}{64} \cdot A_{Octagon}$

After an infinite number of steps, the whole area is colored:

$$\frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \ldots = \frac{3}{4} \cdot \left(1 + \frac{1}{4} + \frac{1}{16} + \ldots\right) = \frac{3}{4} \cdot \frac{1}{1-\frac{1}{4}} = \frac{3}{4} \cdot \frac{1}{\frac{3}{4}} = 1.$$

Illustrations below: Since the radius is divided by 3, the remaining yellow colored area covers one third of the total area of the octagon, i.e. the light blue colored area of the 1st step: $A_1 = \frac{2}{3} \cdot A_{Octagon}$ and further for the purple-colored octagon, which occupies two thirds of the remaining yellow-colored area:

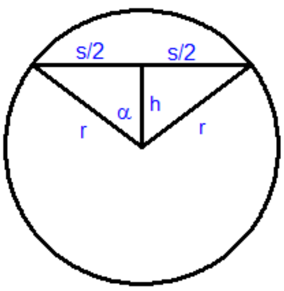
$A_2 = \frac{2}{3} \cdot \frac{1}{3} \cdot A_{Octagon} = \frac{2}{9} \cdot A_{Octagon}$.

Before we start step 3, one third of one third of the octagonal area is still yellow. In step 3, two thirds of it is then coloured pink: $A_3 = \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot A_{Octagon} = \frac{2}{27} \cdot A_{Octagon}$

After an infinite number of steps, the whole area is colored.

✱ A 8.13:

- **Trisection of the circle**



The area of a circular segment can be calculated as difference between the area of the sector and the area of the isosceles triangle below the chord. Because of sin(α) = s/2r and cos(α) = h/r the following applies

$$A_{circularsegment} = \pi \cdot r^2 \cdot \frac{\alpha}{180^\circ} - \frac{1}{2} \cdot s \cdot h = \pi \cdot r^2 \cdot \frac{\alpha}{180^\circ} - r \cdot \sin(\alpha) \cdot r \cdot \cos(\alpha) = r^2 \cdot [\pi \cdot \frac{\alpha}{180^\circ} - \sin(\alpha) \cdot \cos(\alpha)]$$

Since the three colored parts of the circle each have the area 1/3 · π · r², the following equation must be solved: $\pi \cdot \frac{\alpha}{180^\circ} - \sin(\alpha) \cdot \cos(\alpha) = \frac{\pi}{3}$

The solution is 74.64°. From this follows: s ≈ 1.929 · r and h ≈ 0.265 · r. Because of the figure's symmetry. this also applies to the lower circular segment.

The three layers therefore have the heights 0.735 · r and 0.530 · r and 0.735 · r.

(*Control*: the sum is 2r.)

- **Partition of the circle into four layers**

The circle is divided into two halves by a diameter. Here, analogously to above, the equation

$\pi \cdot \frac{\alpha}{180^\circ} - \sin(\alpha) \cdot \cos(\alpha) = \frac{\pi}{4}$ must be solved. The solution is α ≈ 66.17° and from this we get s ≈ 1,830 · r and h ≈ 0,404 · r.

The four layers therefore have the heights 0.596 · r and 0.404 · r and 0.404 · r and 0.596 · r.

- **Partition of the circle into five layers**

Analogously to above, the following equations must be solved

$$\pi \cdot \frac{\alpha}{180^\circ} - \sin(\alpha) \cdot \cos(\alpha) = \frac{\pi}{5} \quad \text{and} \quad \pi \cdot \frac{\alpha}{180^\circ} - \sin(\alpha) \cdot \cos(\alpha) = \frac{2\pi}{5}$$

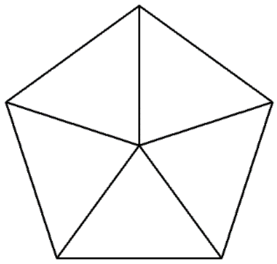
to determine the area of the the two upper layers:

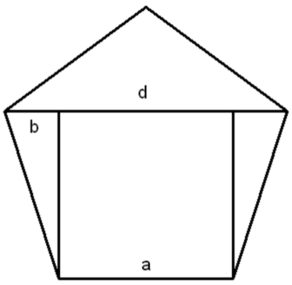
From α ≈ 60.54° it follows: s ≈ 1.74 · r and h ≈ 0.492 · r. And from α ≈ 80.92° it follows: s ≈ 1.97 · r and h ≈ 0.158 · r.

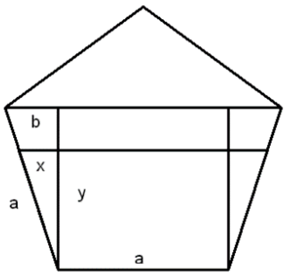
Therefore the five layers of the circle have the heights:

0.508 · r, 0.334 · r, 0.316 · r, 0.334 · r and 0.508 · r.

- **Partition of the regular pentagon**







A regular pentagon with side length a consists of five symmetric triangles. The altitude H of these triangles can be calculated by $\tan(36^\circ) = \frac{a}{2H} \Leftrightarrow H = \frac{a}{2 \cdot \tan(36^\circ)}$, so the area of the regular pentagon is

$$A_{Pentagon} = 5 \cdot \frac{a}{2} \cdot \frac{a}{2 \cdot \tan(36^\circ)} = \frac{5}{4 \cdot \tan(36^\circ)} \cdot a^2 \approx 1.7205 \cdot a^2 .$$

The lower part of the pentagon is a trapezoid formed by three sides of the pentagon and a diagonal d. This diagonal d can be calculated as follows (see central figure):

d = a + 2 · b = a + 2 · a · sin(18°) = a · (1 + 2 · sin(18°)) ≈ 1.618 · a

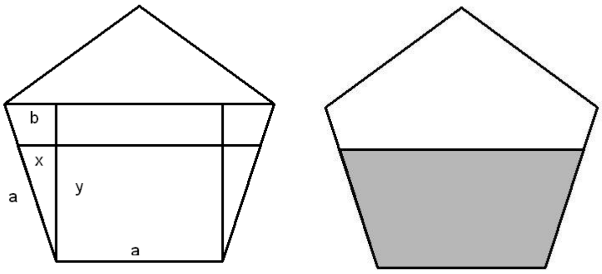
It could also be calculated using d = 2a · sin(54°).

The altitude of the trapezoid is: h = a · cos(18°) ≈ 0.951a; therefore the area of the trapezoid is calculated as follows:

$$A_{trapezoid} = \tfrac{1}{2} \cdot (a+d) \cdot h = \tfrac{1}{2} \cdot (a + a \cdot (1 + 2 \cdot \sin(18°))) \cdot a \cdot \cos(18°)$$
$$= a^2 \cdot \cos(18°) \cdot (1 + \sin(18°)) \approx 1.2449 \cdot a^2$$

The area of the trapezoid is approximately 72.4 % of the total area of the regular pentagon; therefore, we can conclude, that the bisecting line of the pentagon lies *below* the diagonal.

- **Bisection of the pentagon** (is needed above)



If you draw a parallel to the diagonal, a trapezoid is created whose area is calculated as follows:

$$A = a \cdot y + 2 \cdot \frac{1}{2} \cdot x \cdot y = a^2 \cdot k \cdot \cos(18°) + a^2 \cdot k^2 \cdot \sin(18°) \cdot \cos(18°)$$
$$= a^2 \cdot \cos(18°) \cdot k \cdot (1 + k \cdot \sin(18°))$$

The resulting distances can be calculated as follows (according to the intercept theorem):

k = y : h = x : b

We search the value of the ratio k that fulfills the following equation:

$$a^2 \cdot \cos(18°) \cdot k \cdot (1 + k \cdot \sin(18°)) = \frac{1}{2} \cdot \frac{5}{4\tan(36°)} \cdot a^2$$

With the aid of CAS we find the solution: k ≈ 0.737 and with this we get the altitude of the new trapezoid

$y = k \cdot a \cdot \cos(18°) \approx 0.701 \cdot a$

- **Partition of the pentagon into three layers**

In order to determine the lower line for the partition, one must solve the equation

$$a^2 \cdot \cos(18°) \cdot k \cdot (1 + k \cdot \sin(18°)) = \frac{1}{3} \cdot \frac{5}{4\tan(36°)} \cdot a^2$$

From the solution k ≈ 0.520 we get y ≈ 0.495 · a.

Analogously we get the upper line from $a^2 \cdot \cos(18°) \cdot k \cdot (1 + k \cdot \sin(18°)) = \frac{2}{3} \cdot \frac{5}{4\tan(36°)} \cdot a^2$

with the solution k ≈ 0.936 and thus $y \approx 0.890 \cdot a$.

- **Partition of the pentagon into four layers**

In order to determine the lower line for the partition, one must solve the equation

$$a^2 \cdot \cos(18°) \cdot k \cdot (1 + k \cdot \sin(18°)) = \frac{1}{4} \cdot \frac{5}{4\tan(36°)} \cdot a^2$$

From the solution $k \approx 0.402$ we get $y \approx 0.383 \cdot a$.

The second line was calculated below as a bisecting line.

The upper line of the partition lies above the diagonal parallel to the base side.

The isosceles triangle has d as base and legs of side length a.

The area of the triangle results from the area calculated above:

$A_{triangle} = A_{pentagon} – A_{trapezoid} = 1.7205 \cdot a^2 – 1.2449 \cdot a^2 = 0.4756 \cdot a^2$

Since the diagonal has the length $d \approx 1.618 \cdot a$, we can calculate the altitude from the area:
$2 \cdot 0.4756/1.618 \approx 0.588 \cdot a$.

Since the upper quarter of the pentagon has an area of $1.7205 \cdot a^2/4 \approx 0.4301 \cdot a^2$, 90.43 % ($0.4301/0.4756 \approx 0.9043$) of the area above the diagonal is colored yellow.

Since the yellow colored triangle is similar to the triangle above the diagonal, the base of the yellow colored triangle has a length of $\sqrt{0.9043} \cdot 1.618 \cdot a \approx 1.539 \cdot a$ and the altitude a length of $\sqrt{0.9043} \cdot 0.588 \cdot a \approx 0.559 \cdot a$.

- **Partition of the regular hexagon**

A regular hexagon with side length s is consists of six equilateral triangles. In order to divide the hexagon into **three** layers of equal size, a rectangle must be drawn in the middle, the width b of which is just twice the altitude of the triangles of the hexagon and the area A of which is equal to one third of the area of the regular hexagon:

$b = \frac{s}{\tan(30°)}$ and $A = \frac{A_6}{3} = \frac{s^2}{2 \cdot \tan(30°)}$.

The altitude x of the rectangle in the middle can thus be calculated by

$$x = \frac{A}{b} = \frac{s^2}{2 \cdot \tan(30°)} \cdot \frac{\tan(30°)}{s} = \frac{s}{2}$$

To divide a regular hexagon into **four** layers of equal area, one must draw a rectangle in the middle whose width b is just twice the altitude of the above-mentioned triangles of the hexagon and whose area A is equal to a quarter of the area of the regular hexagon:

$b = \frac{s}{\tan(30°)}$ und $A = \frac{A_6}{4} = \frac{3 \cdot s^2}{8 \cdot \tan(30°)}$.

The altitude x of the rectangle in the middle can thus be calculated by

$$x = \frac{A}{b} = \frac{3 \cdot s^2}{8 \cdot \tan(30°)} \cdot \frac{\tan(30°)}{s} = \frac{3}{8} \cdot s$$

To divide the hexagon into **five** layers you get analogously for the altitude x of the rectangle:

$$x = \frac{A}{b} = \frac{3 \cdot s^2}{10 \cdot \tan(30°)} \cdot \frac{\tan(30°)}{s} = \frac{3}{10} \cdot s\ .$$

Chapter 9

The following table contains the representation in the ternal system and in the balanced ternal system for all natural numbers between 1 and 121. For the natural numbers highlighted in green in the table, no compensation is required on the other weighing pan. The number of weights required for weighing can be seen in the last column. For 5 of the 121 numbers (object's weights) considered 1 balance weight is sufficient, for 20 numbers you need 2 balance weights, for 40 numbers 3 balance weights are required, also 4 balance weights for another 40 numbers, and for 16 numbers all 5 weights are required.

Number in the decimal system	Number in the ternal system					Number in the balanced ternal system					Number of balance weights needed
1	0	0	0	0	1	0	0	0	0	1	1
2	0	0	0	0	2	0	0	0	1	-1	2
3	0	0	0	1	0	0	0	0	1	0	1
4	0	0	0	1	1	0	0	0	1	1	2
5	0	0	0	1	2	0	0	1	-1	-1	3
6	0	0	0	2	0	0	0	1	-1	0	2
7	0	0	0	2	1	0	0	1	-1	1	3
8	0	0	0	2	2	0	0	1	0	-1	2
9	0	0	1	0	0	0	0	1	0	0	1
10	0	0	1	0	1	0	0	1	0	1	2
11	0	0	1	0	2	0	0	1	1	-1	3
12	0	0	1	1	0	0	0	1	1	0	2
13	0	0	1	1	1	0	0	1	1	1	3
14	0	0	1	1	2	0	1	-1	-1	-1	4
15	0	0	1	2	0	0	1	-1	-1	0	3
16	0	0	1	2	1	0	1	-1	-1	1	4
17	0	0	1	2	2	0	1	-1	0	-1	3
18	0	0	2	0	0	0	1	-1	0	0	2
19	0	0	2	0	1	0	1	-1	0	1	3
20	0	0	2	0	2	0	1	-1	1	-1	4
21	0	0	2	1	0	0	1	-1	1	0	3
22	0	0	2	1	1	0	1	-1	1	1	4
23	0	0	2	1	2	0	1	0	-1	-1	3
24	0	0	2	2	0	0	1	0	-1	0	2
25	0	0	2	2	1	0	1	0	-1	1	3
26	0	0	2	2	2	0	1	0	0	-1	2
27	0	1	0	0	0	0	1	0	0	0	1
28	0	1	0	0	1	0	1	0	0	1	2
29	0	1	0	0	2	0	1	0	1	-1	3
30	0	1	0	1	0	0	1	0	1	0	2
31	0	1	0	1	1	0	1	0	1	1	3
32	0	1	0	1	2	0	1	1	-1	-1	4
33	0	1	0	2	0	0	1	1	-1	0	3

34	0	1	0	2	1	0	1	1	-1	1	4
35	0	1	0	2	2	0	1	1	0	-1	3
36	0	1	1	0	0	0	1	1	0	0	2
37	0	1	1	0	1	0	1	1	0	1	3
38	0	1	1	0	2	0	1	1	1	-1	4
39	0	1	1	1	0	0	1	1	1	0	3
40	0	1	1	1	1	0	1	1	1	1	4
41	0	1	1	1	2	1	-1	-1	-1	-1	5
42	0	1	1	2	0	1	-1	-1	-1	0	4
43	0	1	1	2	1	1	-1	-1	-1	1	5
44	0	1	1	2	2	1	-1	-1	0	-1	4
45	0	1	2	0	0	1	-1	-1	0	0	3
46	0	1	2	0	1	1	-1	-1	0	1	4
47	0	1	2	0	2	1	-1	-1	1	-1	5
48	0	1	2	1	0	1	-1	-1	1	0	4
49	0	1	2	1	1	1	-1	-1	1	1	5
50	0	1	2	1	2	1	-1	0	-1	-1	4
51	0	1	2	2	0	1	-1	0	-1	0	3
52	0	1	2	2	1	1	-1	0	-1	1	4
53	0	1	2	2	2	1	-1	0	0	-1	3
54	0	2	0	0	0	1	-1	0	0	0	2
55	0	2	0	0	1	1	-1	0	0	1	3
56	0	2	0	0	2	1	-1	0	1	-1	4
57	0	2	0	1	0	1	-1	0	1	0	3
58	0	2	0	1	1	1	-1	0	1	1	4
59	0	2	0	1	2	1	-1	1	-1	-1	5
60	0	2	0	2	0	1	-1	1	-1	0	4
61	0	2	0	2	1	1	-1	1	-1	1	5
62	0	2	0	2	2	1	-1	1	0	-1	4
63	0	2	1	0	0	1	-1	1	0	0	3
64	0	2	1	0	1	1	-1	1	0	1	4
65	0	2	1	0	2	1	-1	1	1	-1	5
66	0	2	1	1	0	1	-1	1	1	0	4
67	0	2	1	1	1	1	-1	1	1	1	5
68	0	2	1	1	2	1	0	-1	-1	-1	4
69	0	2	1	2	0	1	0	-1	-1	0	3
70	0	2	1	2	1	1	0	-1	-1	1	4
71	0	2	1	2	2	1	0	-1	0	-1	3
72	0	2	2	0	0	1	0	-1	0	0	2
73	0	2	2	0	1	1	0	-1	0	1	3
74	0	2	2	0	2	1	0	-1	1	-1	4
75	0	2	2	1	0	1	0	-1	1	0	3
76	0	2	2	1	1	1	0	-1	1	1	4
77	0	2	2	1	2	1	0	0	-1	-1	3

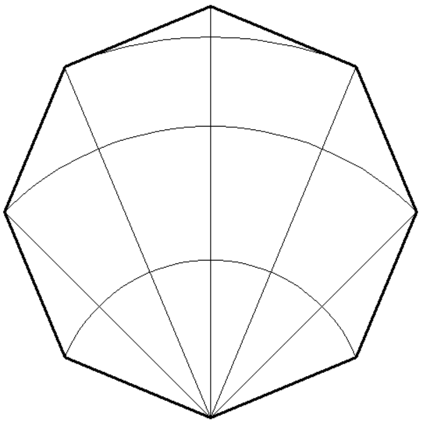
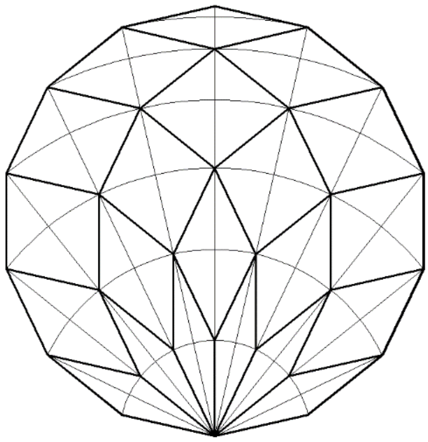
78	0	2	2	2	0	1	0	0	-1	0	2
79	0	2	2	2	1	1	0	0	-1	1	3
80	0	2	2	2	2	1	0	0	0	-1	2
81	1	0	0	0	0	1	0	0	0	0	1
82	1	0	0	0	1	1	0	0	0	1	2
83	1	0	0	0	2	1	0	0	1	-1	3
84	1	0	0	1	0	1	0	0	1	0	2
85	1	0	0	1	1	1	0	0	1	1	3
86	1	0	0	1	2	1	0	1	-1	-1	4
87	1	0	0	2	0	1	0	1	-1	0	3
88	1	0	0	2	1	1	0	1	-1	1	4
89	1	0	0	2	2	1	0	1	0	-1	3
90	1	0	1	0	0	1	0	1	0	0	2
91	1	0	1	0	1	1	0	1	0	1	3
92	1	0	1	0	2	1	0	1	1	-1	4
93	1	0	1	1	0	1	0	1	1	0	3
94	1	0	1	1	1	1	0	1	1	1	4
95	1	0	1	1	2	1	1	-1	-1	-1	5
96	1	0	1	2	0	1	1	-1	-1	0	4
97	1	0	1	2	1	1	1	-1	-1	1	5
98	1	0	1	2	2	1	1	-1	0	-1	4
99	1	0	2	0	0	1	1	-1	0	0	3
100	1	0	2	0	1	1	1	-1	0	1	4
101	1	0	2	0	2	1	1	-1	1	-1	5
102	1	0	2	1	0	1	1	-1	1	0	4
103	1	0	2	1	1	1	1	-1	1	1	5
104	1	0	2	1	2	1	1	0	-1	-1	4
105	1	0	2	2	0	1	1	0	-1	0	3
106	1	0	2	2	1	1	1	0	-1	1	4
107	1	0	2	2	2	1	1	0	0	-1	3
108	1	1	0	0	0	1	1	0	0	0	2
109	1	1	0	0	1	1	1	0	0	1	3
110	1	1	0	0	2	1	1	0	1	-1	4
111	1	1	0	1	0	1	1	0	1	0	3
112	1	1	0	1	1	1	1	0	1	1	4
113	1	1	0	1	2	1	1	1	-1	-1	5
114	1	1	0	2	0	1	1	1	-1	0	4
115	1	1	0	2	1	1	1	1	-1	1	5
116	1	1	0	2	2	1	1	1	0	-1	4
117	1	1	1	0	0	1	1	1	0	0	3
118	1	1	1	0	1	1	1	1	0	1	4
119	1	1	1	0	2	1	1	1	1	-1	5
120	1	1	1	1	0	1	1	1	1	0	4
121	1	1	1	1	1	1	1	1	1	1	5

Chapter 10

✱ A 10.1:

The description can be done as in general terms in section 10.3. The angle depends on the number n.

✱ A 10.2:

✱ A 10.3:

The tessellation of type 1 or type 4 or type 2 or type 5 are obtained again.

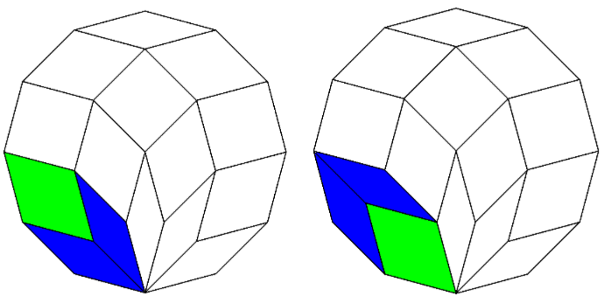
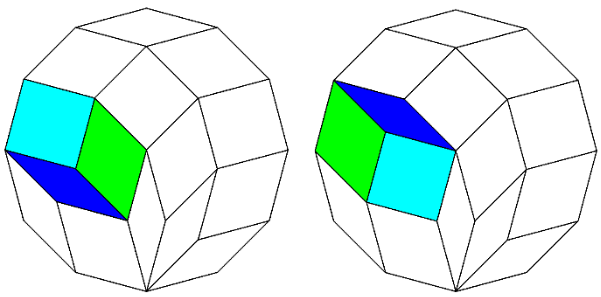
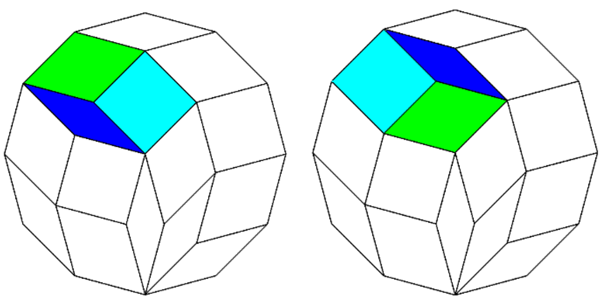
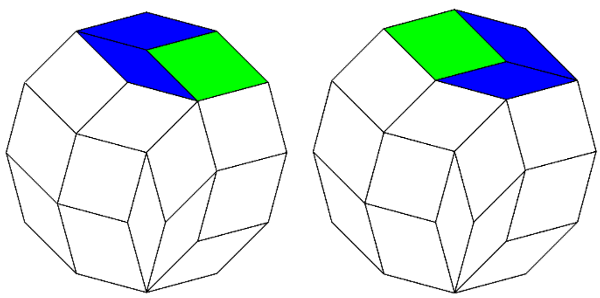
✱ A 10.4:

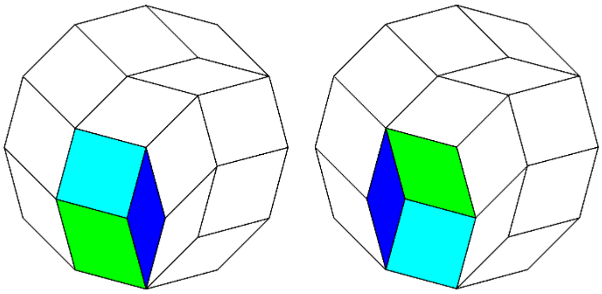
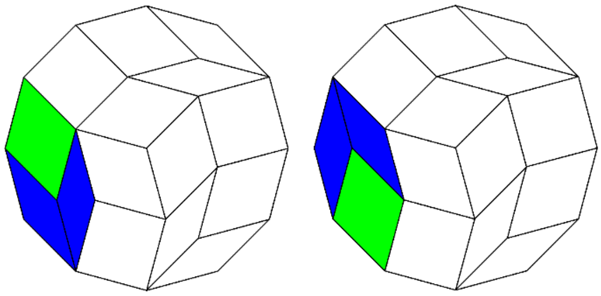
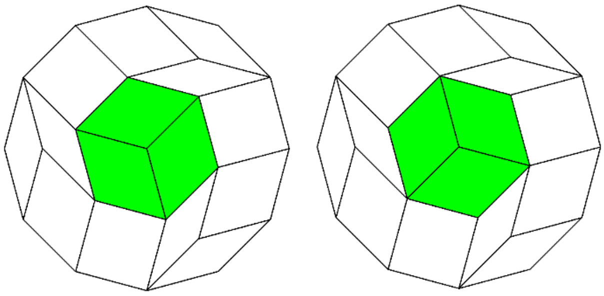
For the tessellation of the regular hexagon, three rhombi of the same type are used; there is no symmetrical partial area (except the area itself) that could be rotated.

Although the regular octagon has symmetrical partial areas that could be rotated, but the figure created after rotation is identical to the original figure.

✱ A 10.5:

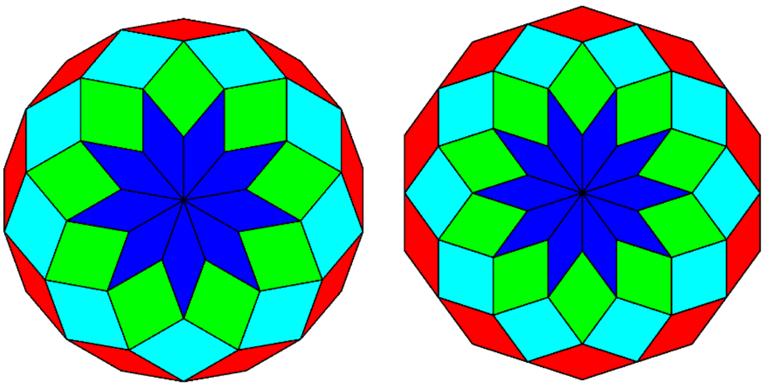
In the following illustrations, it is highlighted in color which areas have been rotated:

from the tessellation in picture 1 to picture 2 	from the tessellation in picture 2 to picture 3 
from the tessellation in picture 3 to picture 4 	from the tessellation in picture 4 to picture 5 

from the tessellation in picture 5 to picture 6 	from the tessellation in picture 6 to picture 7 
from the tessellation in picture 7 to picture 8 	

✱ A 10.6:

If the rhombi are layed out from the centre at an acute angle of 180°/9 = 20° or 180°/10 = 18°, a centrally symmetrical tessellation of a figure is achieved: in the first case it is a regular 18-sided polygon, but in the second picture it is a regular 10-sided polygon.



✱ A 10.7:

The tessellation starts with n rhombi (n = 5, 6, 7, 8) with acute angles of 360°/n. Two diamonds of the same type are then placed on each of these n diamonds (i.e. a total of 2n rhombi) and a further rhombus of the same type fits into the gap between two diamonds. Thus a total of 4n diamonds with an acute angle of 360°/n are required.

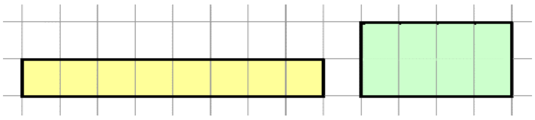
Chapter 11

✱ A 11.1:

To an area of $A = 8$ two rectangles exist

with $p = 8 + 1 + 8 + 1 = 2 \cdot (8 + 1) = 18$ and

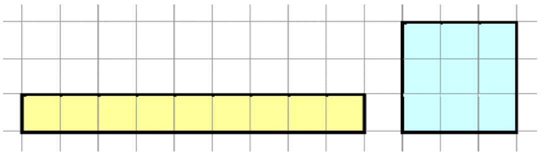
with $p = 4 + 2 + 4 + 2 = 2 \cdot (4 + 2) = 12$.



To an area of $A = 9$ two rectangles exist

with $p = 9 + 1 + 9 + 1 = 2 \cdot (9 + 1) = 20$ and

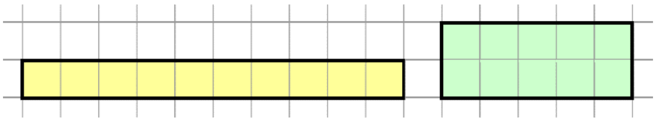
with $p = 3 + 3 + 3 + 3 = 2 \cdot (3 + 3) = 12$.



To an area of $A = 10$ two rectangles exist

with $p = 10 + 1 + 10 + 1 = 2 \cdot (10 + 1) = 22$ and

with $p = 5 + 2 + 5 + 2 = 2 \cdot (5 + 2) = 14$.



✱ A 11.2:

6 can be related to (1 ; 6) and (2 ; 3) as pairs of divisors, the corresponding perimeters are 14 and 10,

8 can be related to (1 ; 8) und (2 ; 4), the corresponding perimeters are 18 and 12 ,

9 can be related to (1 ; 9) und (3 ; 3), the corresponding perimeters are 20 and 12,

10 can be related to (1 ; 10) und (2 ; 5), the corresponding perimeters are 22 and 14.

✱ A 11.3:

Since the additional pairs of divisors are involved here, the restrictions must be noted, since the other cases are already included.

✱ A 11.4:

In the graphic, the possible rectangles that belong to a certain perimeter are represented; highlighted by a square symbol are the rectangles with maximum area that belong to the rectangle perimeters that are divisible by 4, i.e. that they belong to a square.

✱ A 11.5:

A maximum of $(a - 1) \cdot (b - 1)$ unit squares at the corners of an $a \times b$-rectangle. This is best illustrated by the L-shaped figure, which consists of only the first column and the bottom row of the initial rectangle, or by the cross-shaped figure, where the figure consists of $a + b - 1$ unit squares ($= a \cdot b - (a - 1) \cdot (b - 1)$ $= a + b - 1$).

✱ A 11.6:

The following examples show how this can be done:

If the width a is an *odd* number, then one can cut out vertically a maximum of ½ · $(a - 1)$ unit squares, each of which has the height $b - 1$ (amplifying the perimeter by 2 length units each), i.e. the perimeter grows

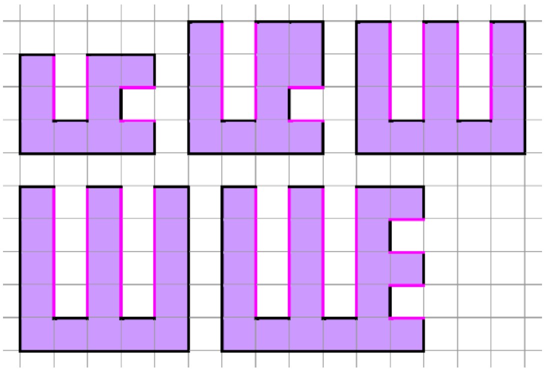
all together: $(a - 1) \cdot (b - 1) = a \cdot b - a - b + 1$.

If the width a is an *even* number, then one can cut out vertically a maximum of $\frac{1}{2} \cdot a - 1$ uni squares, which each have the height $b - 1$ (amplifying the perimeter by 2 length units each), i.e. together $(a - 2) \cdot (b - 1)$, additionally (see the figures on the right) single "cuts" of one unit square are possible, namely for even b these are $\frac{1}{2} \cdot b - 1$ unit squares, amplifying the perimeter by $b - 2$ length units each, for odd b these are $\frac{1}{2} \cdot (b - 1)$ unit squares, amplifying the perimenter by $b - 1$ length units each, i.e. the perimeter increases

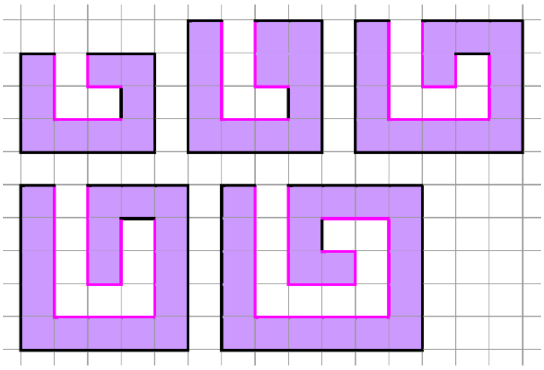
- for even b by a maximum of $(a - 2) \cdot (b - 1) + (b - 2) = a \cdot b - a - b = (a - 1) \cdot (b - 1) - 1$ length units,

- for odd b by a maximum of $(a - 2) \cdot (b - 1) + (b - 1) = (a - 1) \cdot (b - 1) = a \cdot b - a - b + 1$ length units.

In general, the perimeter can be amplified by a maximum of $(a - 1) \cdot (b - 1)$ length units, in case that both a and b are *even* by 1 length unit less.



As an alternative to the vertical cuts, one can also consider which spiral-shaped cuts are possible.

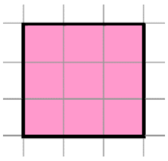


✱ A 11.7:

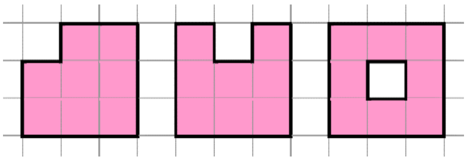
Inner squares can only be drawn if width a and height b are at least 3 length units each. In each case the maximum inner square has the area $(a - 2) \cdot (b - 2)$ square units and the perimeter ist $2 \cdot (a + b - 4)$.

✱ A 11.8:

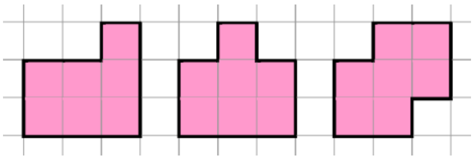
$A = 9$, $p = 12$



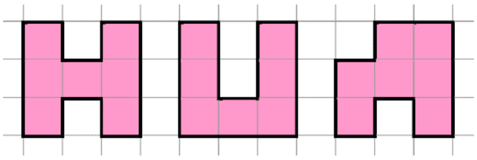
$A = 8$ and (from left to right) $p = 12$, $p = 14$, $p = 16$



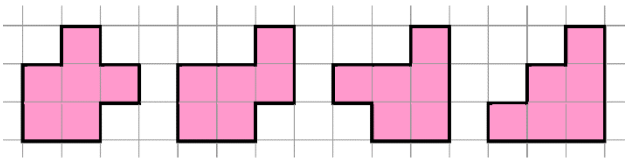
$A = 7$ und $p = 12$



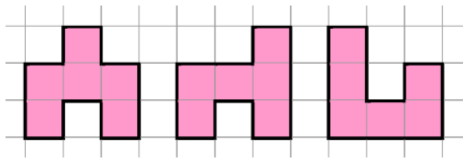
$A = 7$ and (from left to right) $p = 16$, $p = 16$, $p = 14$



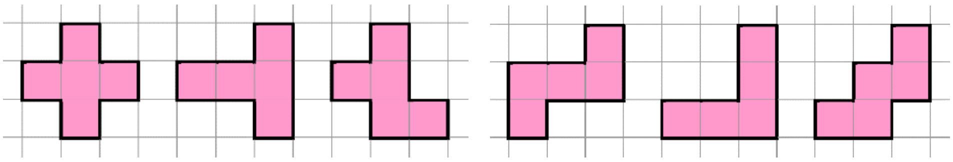
A = 6 and *p* = 12



A = 6 and *p* = 14

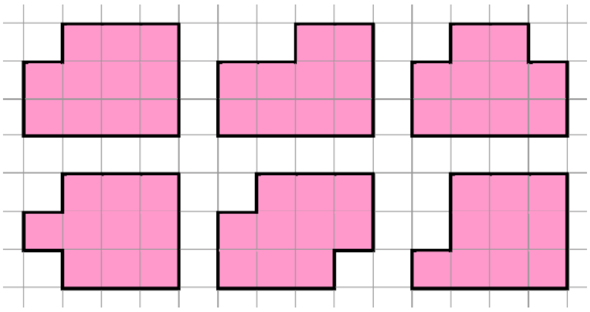


A = 5 and *p* = 12

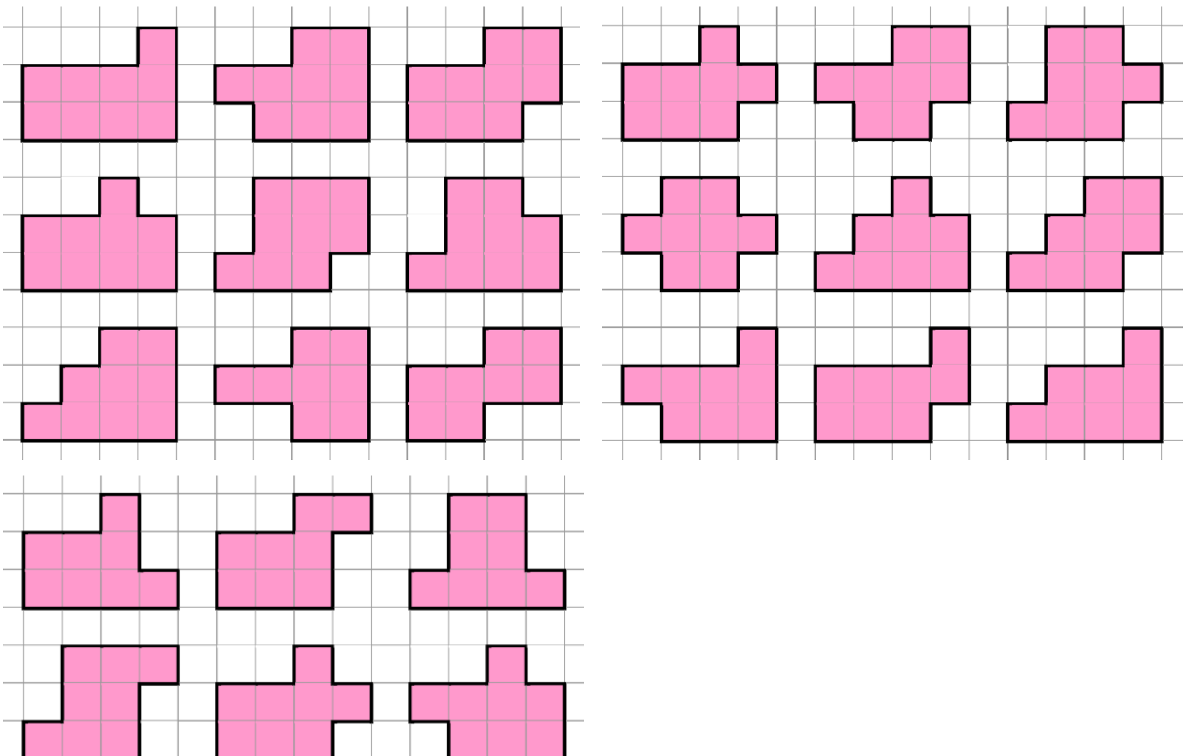


✱ A 11.9:

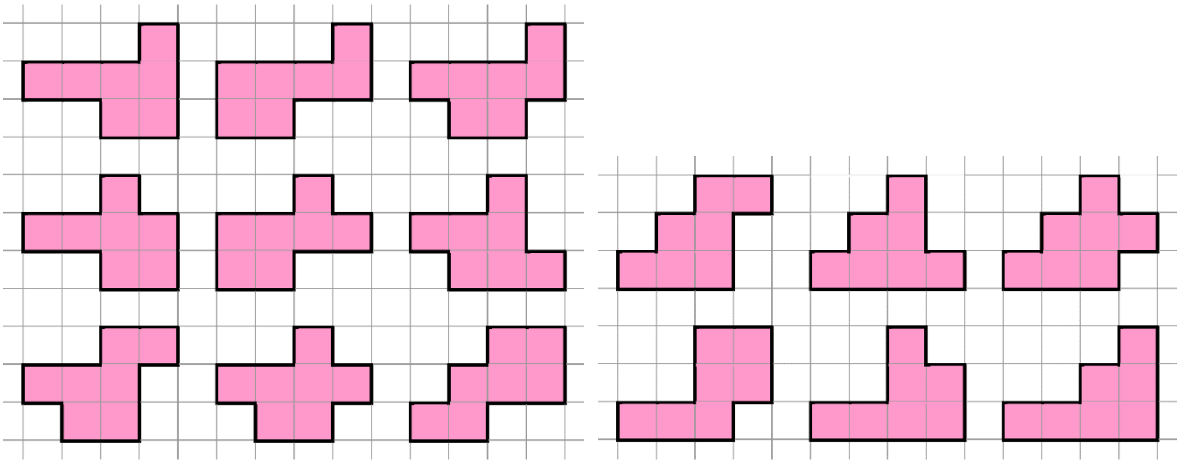
Figures with perimeter *p* = 14 and area *A* = 11 or *A* = 10



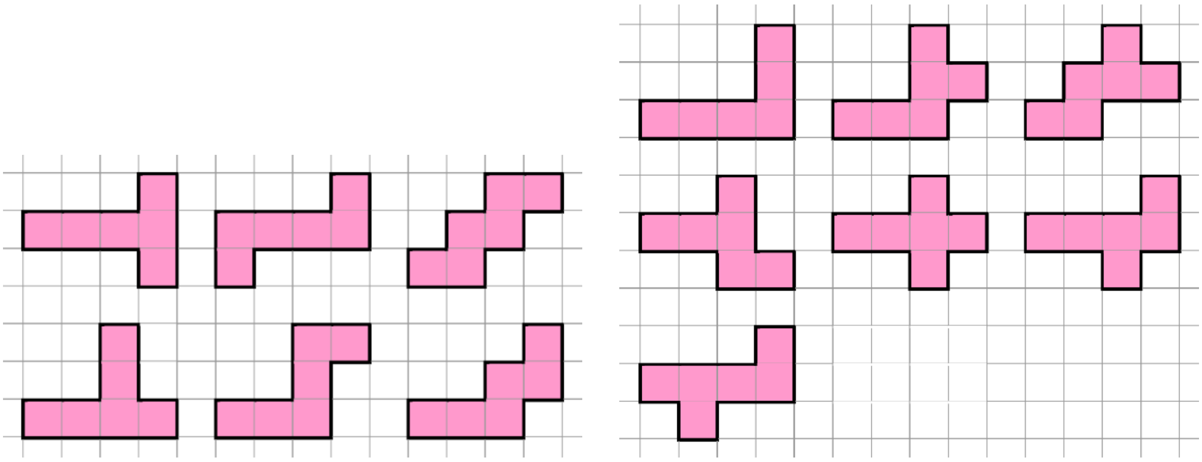
Figures with perimeter *p* = 14 and area *A* = 9 or *A* = 8



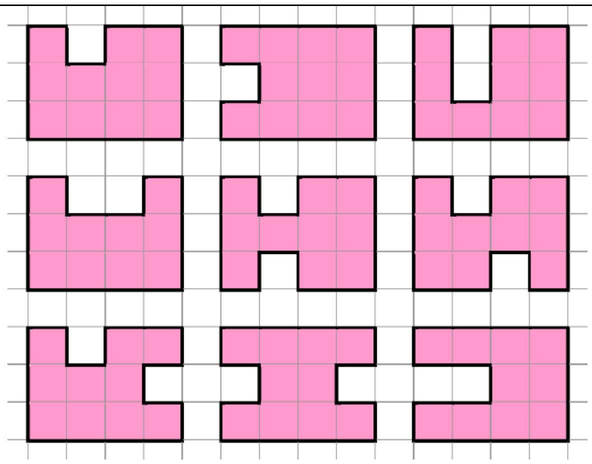
Figures with perimeter p = 14 and area A = 7



Figures with perimeter p = 14 and area A = 6

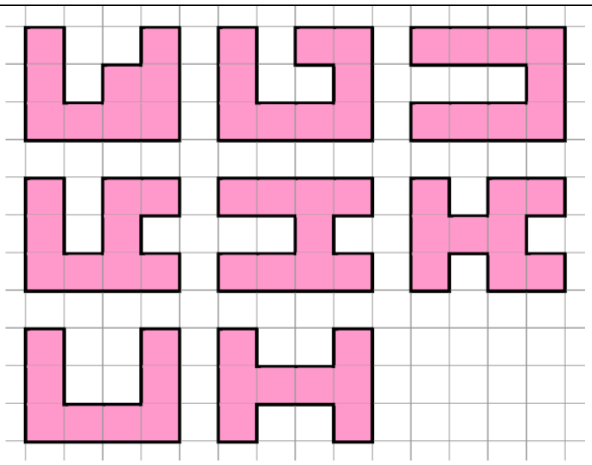


Figures with area A = 11 or A = 10



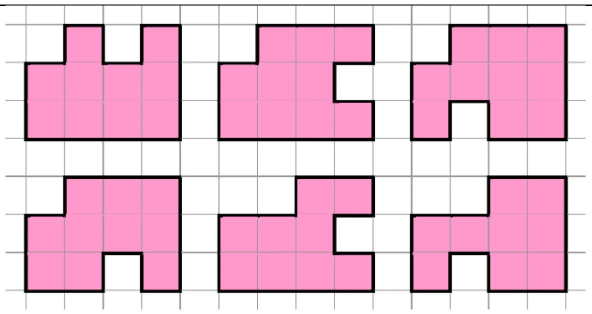
p = 16	p = 16	p = 18
p = 16	p = 18	p = 18
p = 18	p = 18	p = 18

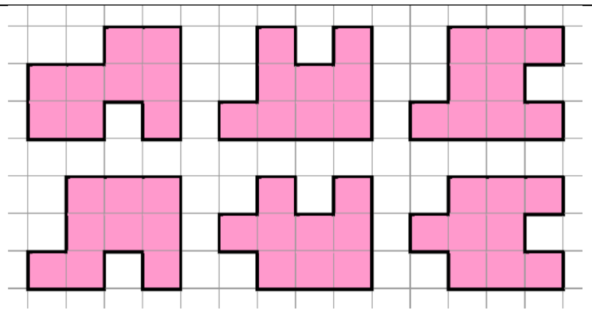
Figures with area $A = 9$ or $A = 8$



$p = 18$	$p = 20$	$p = 20$
$p = 20$	$p = 20$	$p = 20$
$p = 18$	$p = 18$	

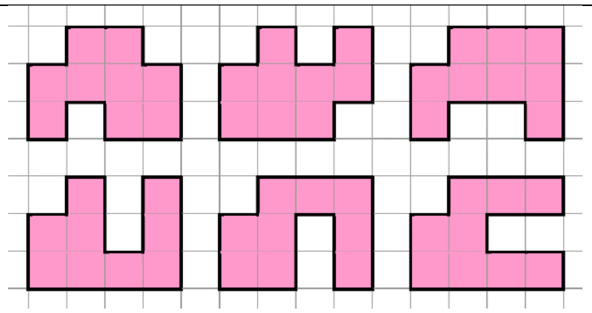
Figures with area $A = 10$ or $A = 9$

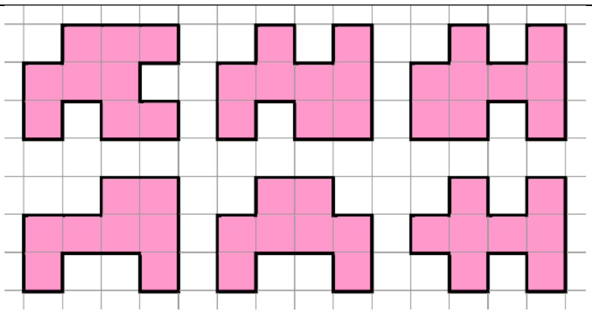




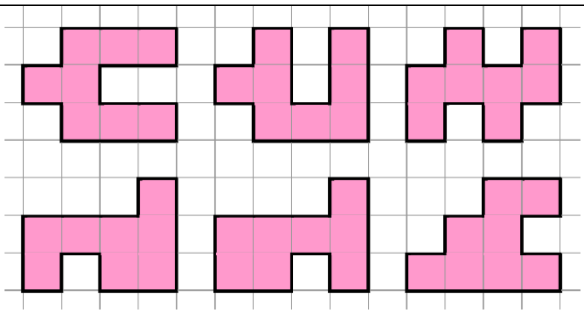
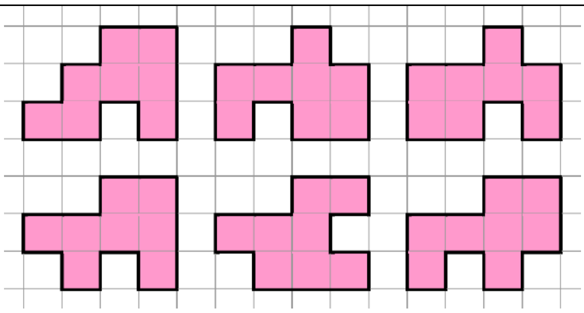
$p = 16$	$p = 16$	$p = 16$
$p = 16$	$p = 16$	$p = 16$
$p = 16$	$p = 16$	$p = 16$
$p = 16$	$p = 16$	$p = 16$

Figures with area $A = 9$ or $A = 8$

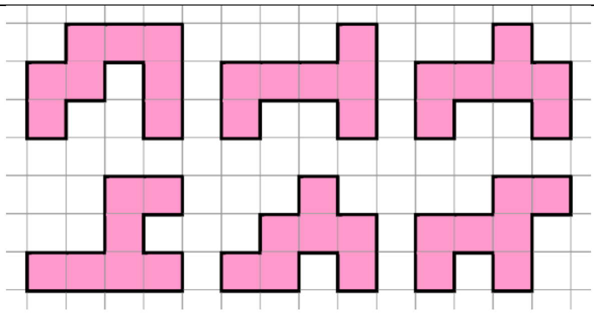




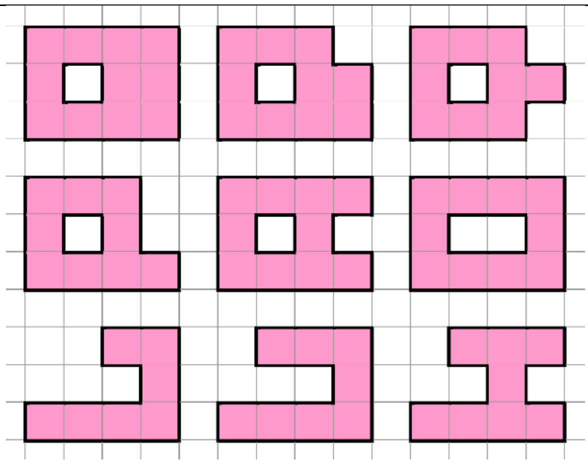
$p = 16$	$p = 16$	$p = 16$
$p = 18$	$p = 18$	$p = 18$
$p = 18$	$p = 18$	$p = 18$
$p = 16$	$p = 16$	$p = 18$

	p = 18	p = 18	p = 18
	p = 16	p = 16	p = 16
	p = 16	p = 16	p = 16
	p = 16	p = 16	p = 16

Figures with area A = 8 or A = 7

	p = 18	p = 16	p = 16
	p = 16	p = 16	p = 16

Figures with area A = 10 or A = 9

	p = 18	p = 18	p = 18
	p = 18	p = 20	p = 20
	p = 16	p = 18	p = 18

✱ A 11.10:

Shown are squares with the areas

$A = 2^2 - 4 \cdot \frac{1}{2} \cdot 1 \cdot 1 = 4 - 2 = 2$; $A = 3^2 - 4 \cdot \frac{1}{2} \cdot 1 \cdot 2 = 9 - 4 = 5$; $A = 4^2 - 4 \cdot \frac{1}{2} \cdot 1 \cdot 3 = 16 - 6 = 10$;

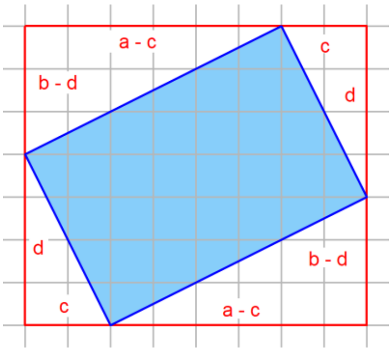
$A = 5^2 - 4 \cdot \frac{1}{2} \cdot 2 \cdot 3 = 25 - 12 = 13$; $A = 5^2 - 4 \cdot \frac{1}{2} \cdot 1 \cdot 4 = 25 - 8 = 17$

These are obtained by cutting off right-angled triangles with side lengths *b* and *a* – *b* at the four vertices of the initial square – all in the same way. For the area then results:
$A = a^2 - 4 \cdot \frac{1}{2} \cdot b \cdot (a - b) = a^2 - 2ab - 2b^2$ with $0 < b \leq a - b < a$

a	*b*	*a* – *b*	*A*
2	1	1	2
3	1	2	5
4	1	3	10
4	2	2	$8 = \frac{1}{2} \cdot a^2$
5	1	4	17
5	2	3	$13 = \frac{1}{2} \cdot (a^2 + 1)$
6	1	5	26
6	2	4	20
6	3	3	$18 = \frac{1}{2} \cdot a^2$
7	1	6	37
7	2	5	29
7	3	4	$25 = \frac{1}{2} \cdot (a^2 + 1)$

a	*b*	*a* – *b*	*A*
8	1	7	50
8	2	6	40
8	3	5	34
8	4	4	$32 = \frac{1}{2} \cdot a^2$
9	1	8	65
9	2	7	53
9	3	6	45
9	4	5	$41 = \frac{1}{2} \cdot (a^2 + 1)$
10	1	9	82
10	2	8	68
10	3	7	58
10	4	6	52
10	5	5	$50 = \frac{1}{2} \cdot a^2$

✱ A 11.11:



The inner rectangles are obtained by starting from the initial rectangles with the side lengths *a* and *b* and then cutting off rectangular triangles with the side lengths *c* and *d* and then a – c and b – d twice. For the area then results: $A = a \cdot b - 2 \cdot \frac{1}{2} \cdot c \cdot d - 2 \cdot \frac{1}{2} \cdot (a - c) \cdot (b - d) = ad + bc - 2cd$
with $0 < c \leq a - c < a$ and $0 < d \leq b - d < b$.

✱ A 11.12:

Durch das Anhängen eines Quadrats erhöht sich die Anzahl der Randpunkte um 2 und es kommt kein innerer Punkt hinzu. Da die Anzahl der Randpunkte mit ½ multipliziert wird, stimmt die Berechnung des Flächeninhalts.

Adding a square increases the number of boundary points by 2 and no interior point is added. Since the number of boundary points is multiplied by ½, the calculation of the area is correct.

✱ A 11.13:

If squares are added or removed at the boundary, the number of boundary points changes by 2 each time; because of the factor ½ for the number of boundary points, this means that the formula applies unchanged.

When k adjacent squares ($k \geq 2$) are added or removed at the boundary, the area is changed by k unit squares.

When removing, the number of interior points decreases by $k + 1$, but the number of boundary points increases by 2:

$A_{new} = (i - k - 1) + ½ \cdot (b + 2) - 1 = (i + ½ \cdot b - 1) - k = A_{old} - k$

When k squares are added, the number of interior points increases by $k - 1$, and the number of boundary points increases by 2:

$A_{new} = (i + k - 1) + ½ \cdot (b + 2) - 1 = (i + ½ \cdot r - 1) + k = A_{old} + k$

✱ A 11.14:

In the first example, one boundary point is added, the number of boundary points remains unchanged. The area increases by ½.

In the second example two boundary points are added, the number of boundary points remains unchanged. The area increases by 1.

In the third example no boundary point is added above, but an interior point is added; accordingly the area increases by 1, according to the formula. The extension of the figure on the right corresponds to the second example.

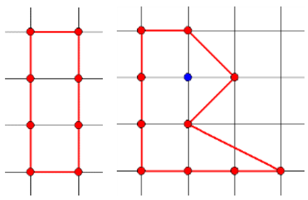
✱ A 11.15:

First example: One diagonally cut at the top on the right: –1 boundary point. Two diagonal cuts on the right: an interior point becomes a boundary point and two boundary points are eliminated.

In the balance the number of interior points decreases by 1 and the number of boundary points by 2, i.e. compared to the initial figure with $A = 2 + ½ \cdot 12 - 1 = 7$ the modified figure has an area of $A = 1 + ½ \cdot 10 - 1 = 5$.

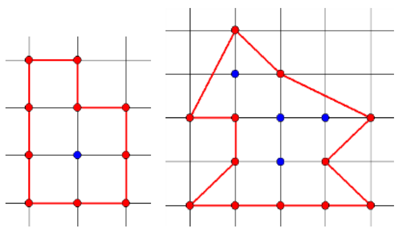
Second example: Due to the diagonally cuts, two boundary points are omitted at the top left, three boundary points at the top right; one boundary point is omitted at the bottom right, but an inner point becomes an boundary point. One boundary point is omitted at the bottom left. Balance: plus 6 boundary points and one interior point less: $A = 5 + ½ \cdot 18 - 1 = 13$, thus we have $A = 4 + ½ \cdot 12 - 1 = 9$.

✱ A 11.16:



1 boundary point becomes an interior point, the number of boundary points increases by 3.

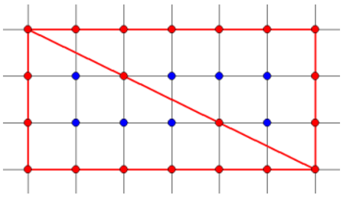
$A_{left} = 0 + ½ \cdot 8 - 1 = 3$; $A_{right} = 1 + ½ \cdot 10 - 1 = 5$



3 boundary poits become interior points, the number of boundary increases by 5.

$A_{left} = 1 + ½ \cdot 10 - 1 = 5$; $A_{right} = 4 + ½ \cdot 12 - 1 = 9$

✱ A 11.17:



If a right-angled triangle is doubled to a rectangle, then the number of interior points is not only doubled, but also k boundary points are added, which on the diagonal:

$i_{rectangle} = 2 \cdot i_{triangle} + k$

For the number of boundary points applies:

$b_{rectangle} = 2 \cdot (b_{triangle} – 1 – k)$

For a rectangle, the correctness of formula (11.2) was proven. Therefore we have

$A_{rectangle} = i_{rectangle} + ½ \cdot b_{rectangle} – 1$

Therefore we get for the triangle which has half of its area:

$A_{trangle} = ½ \cdot A_{rectangle} = ½ \cdot i_{rectangle} + ¼ \cdot b_{rectangle} – ½$

$= i_{triangle} + ½ \cdot k + ½ \cdot r_{triangle} – ½ – ½ \cdot k – ½$

$= i_{triangle} + ½ \cdot b_{riangle} – 1$

✱ A 11.18:

Figure 1: $b_1 = 8$, $i_1 = 1$, $A_1 = 1 + ½ \cdot 8 – 1 = 4$ and $b_2 =14$, $i_2 = 4$, $A_2 = 4 + ½ \cdot 14 – 1 = 10$

2 boundary points coincide, 1 common boundary point becomes an interior point.

$b = b_1 + b_2 – 4 = 18$, $i = i_1 + i_2 + 1 = 6$, $A = 6 + ½ \cdot 18 – 1 = 14$.

Figure 2: $b_1 = 6$, $i_1 = 1$, $A_1 = 1 + ½ \cdot 6 – 1 = 3$ and $b_2 =14$, $i_2 = 4$, $A_2 = 4 + ½ \cdot 14 – 1 = 10$

2 boundary points coincide, 2 common boundary point become interior points.

$b = b_1 + b_2 – 6 = 14$, $i = i_1 + i_2 + 2 = 7$, $A = 7 + ½ \cdot 14 – 1 = 13$.

Figure 3: $b_1 = 6$, $i_1 = 1$, $A_1 = 1 + ½ \cdot 6 – 1 = 3$ and $b_2 = 8$, $i_2 = 2$, $A_2 = 2 + ½ \cdot 8 – 1 = 5$

2 boundary points coincide, 2 common boundary point become interior points.

$b = b_1 + b_2 – 6 = 8$, $i = i_1 + i_2 + 2 = 5$, $A = 5 + ½ \cdot 8 – 1 = 8$.

✱ A 11.19:

Figure 1: The area of the figure is 15.5 – 2 = 13.5. The area of the figure inside can be calculated using Pick's theorem: $b_2 = 4$, $i_2 = 1$, $A_2 = 1 + ½ \cdot 4 - 1 = 2$.

The outer figure has $b_1 = 11$ boundary points and $i_1 = 6$ interior points, so if we apply Pick's theorem we would get: $A_1 = 6 + ½ \cdot 11 - 1 = 10.5$, but the 4 boundary points and the one interior point of the inner figure are not considered, which would be all inner points of the outer figure. So you would have to consider them as interior points first. Then the Pick's theorem would be applicable.

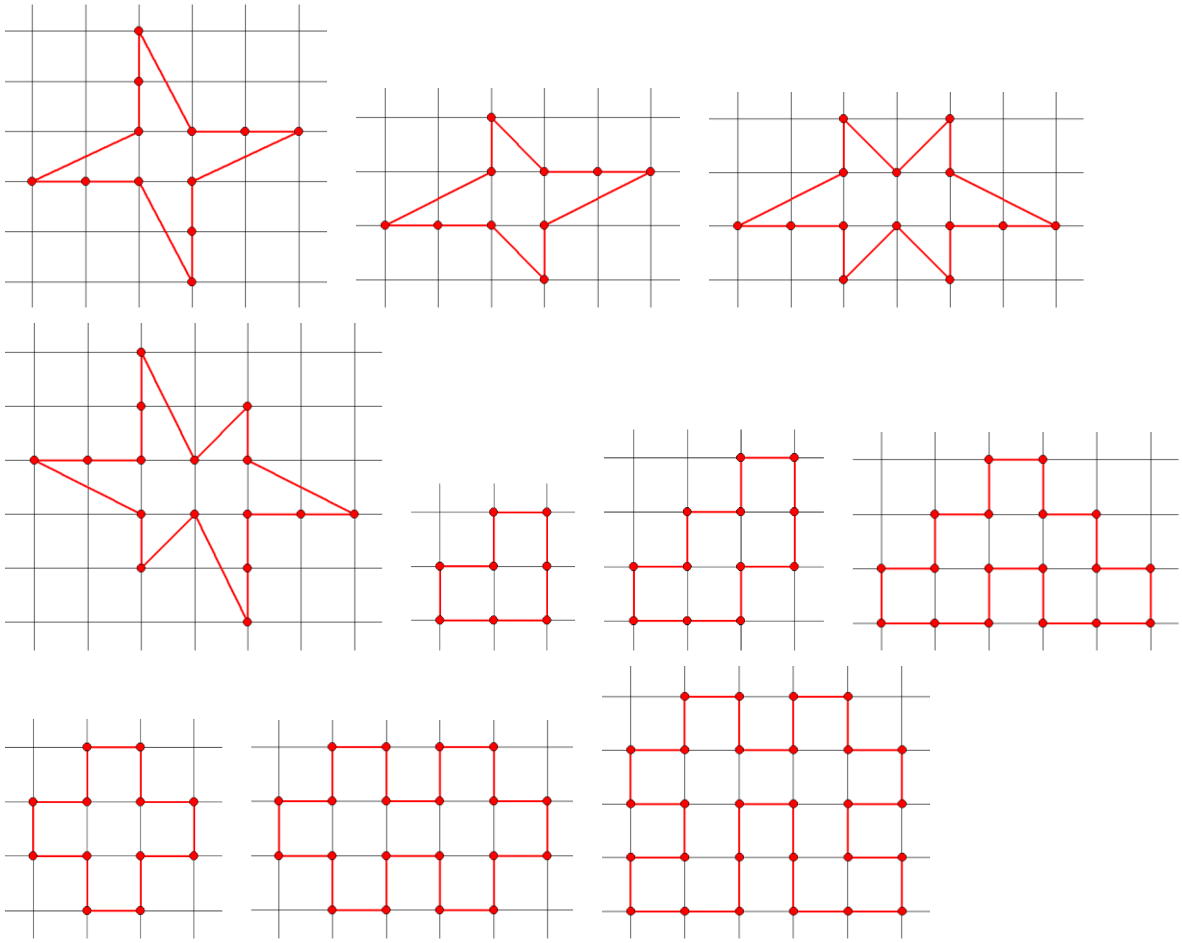
Figure 2: The areas of the two partial figures can be calculated using Pick's theorem:

on the left: $A_1 = 4 + ½ \cdot 6 - 1 = 6$, on the right: $A_2 = 2 + ½ \cdot 4 - 1 = 3$. The area of the whole figure is 9. This is only obtained using Pick's theorem only can be applied if the common point of the two partial figures is counted twice.

Figure 3: The figure has an area of 6.25 + 2.75 = 9. Counting the boundary points (11) and the interior points (6), this would result in an area of $A = 6 + ½ \cdot 11 – 1 = 10.5$. Pick's theorem is therefore not applicable. (If the intersection point of the border lines were a grid point, you could argue as in Figure 2).

✱ A 11.20:

Variation of the given examples:



✱ A 11.21:

(1) Figure on the left: A = 10; Figure on the right: A = 12

(2) Figure on the left: b = 10, i = 1, $A_T = 2 \cdot i + b - 2 = 2 + 10 - 2 = 10$;

Figure on the right: b = 8, i = 3, $A_T = 2 \cdot i + b - 2 = 6 + 8 - 2 = 12$

(3) The formula for a square grid $A_S = i + \frac{1}{2} \cdot b - 1$ and the formula for a triangle grid $A_T = 2 \cdot i + b - 2$ differ only by the factor 2: $A_T = 2 \cdot i + b - 2 = 2 \cdot (i + \frac{1}{2} b - 1) = 2 \cdot A_S$.

This is plausible, because if you look at a rhombus grid instead of a square grid, nothing changes in the formula for the area, in which only the boundary points and the interior points are counted – the area of the figure on the triangular grid is therefore twice the area on the square grid or the rhombus grid.

✱ A 11.22:

(1) $i_H = 1$, $i_T = 3$, $b_H = 12$, $b_T = 0$: $A_H = \frac{1}{3} \cdot (3+1) + \frac{1}{6} \cdot (0+12) - \frac{1}{3} = \frac{4}{3} + 2 - \frac{1}{3} = 3$

(2) $i_H = 1$, $i_T = 0$, $b_H = 8$, $b_T = 1$: $A_H = \frac{1}{3} \cdot (0+1) + \frac{1}{6} \cdot (1+8) - \frac{1}{3} = \frac{1}{3} + \frac{9}{6} - \frac{1}{3} = 1{,}5$

(3) The formula results from the formula for the triangular grid and subsequent division by 6:
$A_H = \frac{1}{6} \cdot (2 \cdot i + b - 2) = \frac{1}{3} \cdot (i_T + i_H) + \frac{1}{6} \cdot (r_T + r_H) - \frac{1}{3}$.

Chapter 12

✱ A 12.1:

The fact that the two dice are not distinguishable does not change the fact that they are two different dice, so it makes a difference whether, for example, the outcome (2,3) or the outcome (3,2) occurs.

✱ A 12.2:

(1) 3 possible combinations exist each for 4 as sum of the spots and for the sum 10; 4 possible combinations exist for the sum 5 and for the sum 9; 5 possible combinations exist for the sum 6 and for the sum 8; 6 possible combinations exist for the sum 7.

1. fair rule: *You win if the sum of the spots is 4, 5, 6 or 7*.
2. fair rule: *You win if the sum of the spots is4, 5, 8 or 7*.
3. fair rule: *You win if the sum of the spots is4, 9, 6 or 7*.
4. fair rule: *You win if the sum of the spots is4, 9, 8 or 7.*
5. fair rule: *You win if the sum of the spots is10, 5, 6 or 7*.
6. fair rule: *You win if the sum of the spots is10, 5, 8 or 7*.
7. fair rule: *You win if the sum of the spots is10, 9, 6 or 7*.
8. fair rule: *You win if the sum of the spots is10, 9, 8 or 7*.

(2) In total there are 44 different fair game rules because you can get the sum 18 from 5 summands as follows:

18 = 1+1+5+5+6 (1 game rule)

Reason: There is 1 possible combination for 2 as sum of the spots and for the sum 12; 5 possible combinations for the sum 6 and for the sum 8; 6 possible combinations for the sum 7. Therefore a fair game rule could be: *You win if the sum of the spots is 2,12, 6, 8 oder 7*.

18 = 1+2+4+5+6 (16 different game rules)

18 = 1+3+3+5+6 (8 different game rules)

18 = 1+3+4+4+6 (4 different game rules)

18 = 1+3+4+5+5 (8 different game rules)

18 = 2+2+3+5+6 (4 different game rules)

18 = 2+2+4+4+6 (1 game rule)

18 = 2+2+4+5+5 (2 different game rules)

Reason: 2 possible combinations for 3 as sum of the spots and for the sum 11; 4 possible combinations for the sum 5 and for the sum 9; 5 possible combinations for the sum 6 and for the sum 8. Therefore fair game rules could be: *You win if the sum of the spots is 3, 11, 5, 6 or 8*. or *You win if the sum of the spots is 3, 11, 9, 6 oder 8.*

18 = 2+3+3+5+5 (2 different game rules)

18 = 2+3+4+4+5 (8 different game rules)

(3) In order to determine fair game rules for three or four players, one must consider how 12 and 9 as the number of combinations can be obtained. Here are some examples ...

for fair rules of the game for three players:

A wins with the following sum of spots	B wins with the following sum of spots	C wins with the following sum of spots
4, 5 or 6 (3+4+5 = 12 combinations)	8, 9 or 10 (5+4+3 = 12 combinations)	2, 3, 7, 11 or 12 (1+2+6+2+1 = 12 combinations)
2, 3, 4 or 7 (1+2+3+6 = 12 combinations)	5, 6 or 10 (4+5+3 = 12 combinations)	8, 9, 11 or 12 (5+4+2+1 = 12 combinations)
2, 4, 5 or 9 (1+3+4+4 = 12 combinations)	3, 6 or 8 (2+5+5 = 12 combinations)	7, 10, 11 or 12 (6+3+2+1 = 12 combinations)

for fair rules of the game for four players:

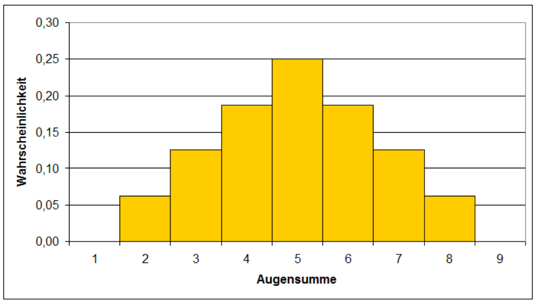
A wins with the following sum of spots	B wins with the following sum of spots	C wins with the following sum of spots	D wins with the following sum of spots
4 or 7 (3+6 = 9 combinations)	5 or 6 (4+5 = 9 combinations)	8 or 9 (5+4 = 9 combinations)	2, 3, 10, 11 or 12 (1+2+3+2+1 = 9 combinations)
3, 4 or 5 (2+3+4 = 9 combinations)	7, 11 or 12 (6+2+1 = 9 combinations)	6 or 9 (5+4 = 9 combinations)	2, 8 or 10 (1+5+3 = 9 combinations)

✱ A 12.3:

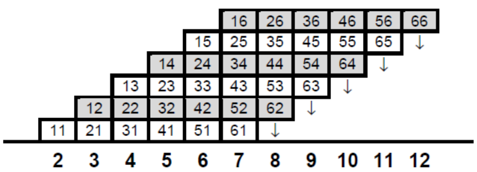
Table of combinations when rolling two tetrahedra

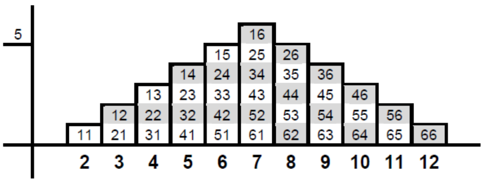
	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

Histogram for the probability distribution when rolling two tetrahedra



Other way of illustration:





(2) Generating function for rolling one tetrahedron: $f(x) = \frac{1}{4} \cdot \left(1x^1 + 1x^2 + 1x^3 + 1x^4\right)$

Factorizing the polynom: $f(x) = \frac{1}{4} \cdot x \cdot (1 + x) \cdot (1 + x^2)$

Therefore we have two possibilities to combine the factors:

$f(x) = \frac{1}{2} \cdot (x^1 + x^2) \cdot \frac{1}{2} \cdot (x^0 + x^2)$ und $f(x) = \frac{1}{2} \cdot (x^0 + x^1) \cdot \frac{1}{2} \cdot (x^1 + x^3)$

Thus we can use wheels of fortune (wheel-of-2) with the numbers (1, 2) and (0, 2) or (0, 1) and (1, 3), as also can be seen from the following combination tables:

	0	2
1	1	3
2	2	4

	1	3
0	1	3
1	2	4

(3)

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

generating function for the sum of spots:

$$f^2(x) = \tfrac{1}{4}\cdot\left[1x^1 + 1x^2 + 1x^3 + 1x^4\right]\cdot\tfrac{1}{4}\cdot\left[1x^1 + 1x^2 + 1x^3 + 1x^4\right]$$
$$= \tfrac{1}{16}\cdot\left(1x^2 + 2x^3 + 3x^4 + 4x^5 + 3x^6 + 2x^7 + 1x^8\right)$$

	1	3	3	5
1	2	4	4	6
2	3	5	5	7
2	3	5	5	7
3	4	6	6	8

alternative labeling of the two tetrahedra
(1, 2, 2, 3) and (1, 3, 3, 5)

corresponding factorization:

$$f^2(x) = \tfrac{1}{4}\cdot\left[1x^1 + 2x^2 + 1x^3\right]\cdot\tfrac{1}{4}\cdot\left[1x^1 + 2x^3 + 1x^5\right]$$
$$= \tfrac{1}{16}\cdot\left(1x^2 + 2x^3 + 3x^4 + 4x^5 + 3x^6 + 2x^7 + 1x^8\right)$$

	1	2	3	3	4	4	5	6
1	2	3	4	4	5	5	6	7
2	3	4	5	5	6	6	7	8

combination of a wheel-of-8 and a wheel-of-2
(1, 2, 3, 3, 4, 4, 5, 6) and (1, 2)

corresponding factorization:

$$f^2(x) = \tfrac{1}{8}\cdot\left[1x^1 + 1x^2 + 2x^3 + 2x^4 + 1x^5 + 1x^6\right]\cdot\tfrac{1}{2}\cdot\left[1x^1 + 1x^2\right]$$
$$= \tfrac{1}{16}\cdot\left(1x^2 + 2x^3 + 3x^4 + 4x^5 + 3x^6 + 2x^7 + 1x^8\right)$$

	1	2	2	3	3	4	4	5
1	2	3	3	4	4	5	5	6
3	4	5	5	6	6	7	7	8

combination of a wheel-of-8 and a wheel-of-2
(1, 2, 2, 3, 3, 4, 4, 5) and (1, 3)

corresponding factorization:

$$f^2(x) = \tfrac{1}{8}\cdot\left[1x^1 + 2x^2 + 2x^3 + 2x^4 + 1x^5\right]\cdot\tfrac{1}{2}\cdot\left[1x^1 + 1x^3\right]$$
$$= \tfrac{1}{16}\cdot\left(1x^2 + 2x^3 + 3x^4 + 4x^5 + 3x^6 + 2x^7 + 1x^8\right)$$

(4) $f'(x) = \frac{1}{4}\cdot(1 + 2x + 3x^2 + 4x^3)$; $f''(x) = \frac{1}{4}\cdot(2 + 6x + 12x^2)$, therefore

$f'(1) = \frac{1}{4}\cdot(1 + 2 + 3 + 4) = 2.5 = \mu$; $f''(1) = \frac{1}{4}\cdot(2 + 6 + 12) = 5$, thus $\sigma^2 = 2{,}5 + 5 - 2{,}5^2 = 1.25$

✱ A 12.4:

The generating function is given by

$$f(x) = \tfrac{1}{4}\cdot\left(1x^1 + 1x^2 + 1x^3 + 1x^4\right)\cdot\tfrac{1}{6}\cdot\left(1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6\right)$$
$$= \tfrac{1}{24}\cdot\left(1x^2 + 2x^3 + 3x^4 + 4x^5 + 4x^6 + 4x^7 + 3x^8 + 2x^9 + 1x^{10}\right)$$

The alternative random devices result from the different decompositions of the generating functions, whereby a wheel-of-2 could be replaced by a coin, an wheel-of-8 by a regular octahedron and a wheel-of-12 by a regular dodecahedron:

$$\tfrac{1}{4}\cdot\left(1x^1+1x^2+1x^3+1x^4\right)\cdot\tfrac{1}{6}\cdot\left(1x^1+1x^2+1x^3+1x^4+1x^5+1x^6\right)$$
$$=\tfrac{1}{2}\cdot\left[1x+1x^2\right]\cdot\tfrac{1}{12}\cdot\left[1x^1+1x^2+2x^3+2x^4+2x^5+2x^6+1x^7+1x^8\right]$$
$$=\tfrac{1}{2}\cdot\left[1x+1x^3\right]\cdot\tfrac{1}{12}\cdot\left[1x^1+2x^2+2x^3+2x^4+2x^5+2x^6+1x^7\right]$$
$$=\tfrac{1}{2}\cdot\left[1x^1+1x^4\right]\cdot\tfrac{1}{12}\cdot\left[1x^1+2x^2+3x^3+3x^4+2x^5+1x^6\right]$$
$$=\tfrac{1}{8}\cdot\left[1x^1+1x^2+1x^3+2x^4+1x^5+1x^6+1x^7\right]\cdot\tfrac{1}{3}\cdot\left[1x+1x^2+1x^3\right]$$
$$=\tfrac{1}{8}\cdot\left[1x^1+2x^2+2x^3+2x^4+1x^5\right]\cdot\tfrac{1}{3}\cdot\left[1x^1+1x^3+1x^5\right]$$
$$=\tfrac{1}{4}\cdot\left[1x+2x^2+1x^3\right]\cdot\tfrac{1}{6}\cdot\left[1x^1+2x^3+2x^5+1x^7\right]$$
$$=\tfrac{1}{4}\cdot\left[1x^1+1x^2+1x^4+1x^5\right]\cdot\tfrac{1}{6}\cdot\left[1x^1+1x^2+2x^3+1x^4+1x^5\right]$$
$$=\tfrac{1}{4}\cdot\left[1x^1+1x^3+1x^4+1x^6\right]\cdot\tfrac{1}{6}\cdot\left[1x^1+2x^2+2x^3+1x^4\right]$$

Random device 1	Labeling	Random device 2	Labeling
coin	(1, 2)	Dodecahedron	(1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 8)
coin	(1, 3)	Dodecahedron	(1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7)
coin	(1, 4)	Dodecahedron	(1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 6)
wheel-of-3	(1, 2, 3)	Octahedron	(1, 2, 3, 4, 4, 5, 6, 7)
wheel-of-3	(1, 3, 5)	Octahedron	(1, 2, 2, 3, 3, 4, 4, 5)
Tetrahedron	(1, 2, 2, 3)	Hexahedron	(1, 3, 3, 5, 5, 7)
Tetrahedron	(1, 2, 4, 5)	Hexahedron	(1, 2, 3, 3, 4, 5)
Tetrahedron	(1, 3, 4, 6)	Hexahedron	(1, 2, 2, 3, 3, 4)

✱ A 12.5:

- Rolling 10 regular hexahedra: $\mu = 10 \cdot 3{,}5 = 35$; $\sigma^2 = 10 \cdot 35/12 \approx 29{,}17$; $\sigma \approx 5{,}40$

– in about two thirds of the experiments in the interval between 30 and 40 (exactly: 69.1 %)

– with a probability of about 90 % in the interval between 26 and 44 (exactly: 92.1 %)

– with a probability of about 95% in the interval between 25 and 45 (exactly: 94.8%)

– with a probability of about 99 % in the interval between 21 and 49 (exactly: 99.3 %)

- Rolling 10 regular octahedra: $\mu = 10 \cdot 4{,}5 = 45$; $\sigma^2 = 10 \cdot 63/12 = 52{,}5$; $\sigma \approx 7{,}25$

– in about two thirds of the experiments in the interval between 38 and 51 (exactly: 66.5 %)

– with a probability of about 90 % in the interval between 33 and 57 (exactly: 91.5 %)

– with a probability of about 95 % in the interval between 31 and 59 (exactly: 95.4 %)

– with a probability of about 99 % in the interval between 26 and 64 (exactly: 99.3 %)

Chapter 13

✱ A 13.1:

Area of the partial figure:

$2 \cdot (8^2 + ½ \cdot 8 \cdot 13 + ½ \cdot 5 \cdot 8) = 272$

$2 \cdot (8 \cdot 21 + ½ \cdot 8 \cdot 13 + ½ \cdot 13 \cdot 21) = 713$

Total area:

$13 \cdot 21 = 273$

$21 \cdot 34 = 714$

In both examples the white band has the area 1. The ratio of the partial area is 0.37 % and 0.14 % respectively.

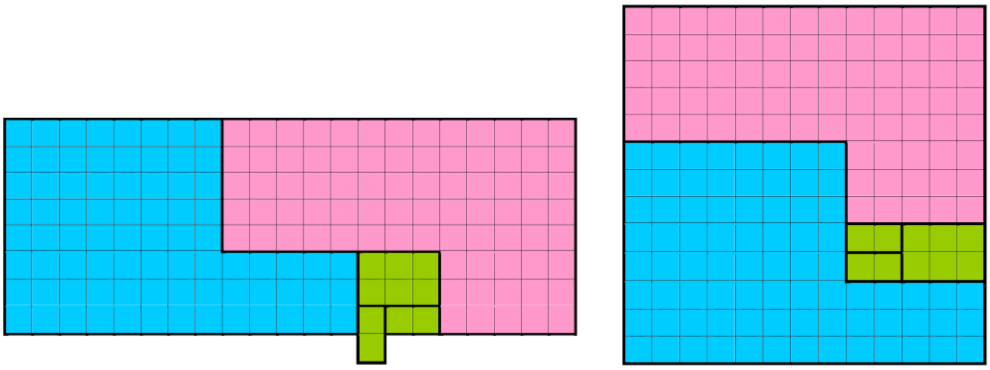
✱ A 13.2:

In the second figure, the green triangle has a gradient of $\tan^{-1}(8/13) \approx 31.61°$, the light blue triangle of $\tan^{-1}(5/8) \approx 32.01°$. Even this difference of 0.40° is hardly noticeable!

The whole figure is seen as a triangle, whose angle is given by the ratio 13:21: $\tan^{-1}(13/21) \approx 31.76°$.

In the third figure, the green triangle has a gradient of $\tan^{-1}(5/8) \approx 32.01°$, the light blue triangle has a gradient of $\tan^{-1}(3/5) \approx 30.96°$. This difference of 1.05° can perhaps be discovered! The whole figure is seen as a triangle whose angle is given by the ratio 8:13: $\tan^{-1}(8/13) \approx 31.61°$.

✱ A 13.3:



✱ A 13.4:

In puzzle 3 the light blue triangle has sides with the lengths 5 and 3; such a triangle does not appear here in the figure (here a side has the length 3 and the hypotenuse the length 5). The green triangle in puzzle 3 has sides with the lengths 5 and 8; here the hypotenuse has the length 8 and a side the length 5. This also applies accordingly to the illustration from puzzle 1.

In the figures considered here, almost similar triangles occur in which the ratio of the length of a side to the length of the hypotenuse are adjacent Fibonacci numbers; the corresponding angles can be calculated using the inverse function of the sine:

$$\sin^{-1}\left(\frac{3}{5}\right) \approx 36.87° \;;\; \sin^{-1}\left(\frac{5}{8}\right) \approx 38.68° \;;\; \sin^{-1}\left(\frac{8}{13}\right) \approx 37.98° ;\; \sin^{-1}\left(\frac{13}{21}\right) \approx 38.25° ;\; \ldots$$

✱ A 13.5:

$$\frac{f_n}{f_{n+2}} = \frac{f_n}{f_n + f_{n+1}} = \frac{1}{\frac{f_n + f_{n+1}}{f_n}} = \frac{1}{1 + \frac{f_{n+1}}{f_n}} \quad \to \quad \frac{1}{1 + \frac{2}{\sqrt{5}-1}} = \frac{1}{\frac{\sqrt{5}+1}{\sqrt{5}-1}} = \frac{\sqrt{5}-1}{\sqrt{5}+1} \approx 0.382 ;$$

$$\tan^{-1}\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right) = 20.905157...°$$

✱ A 13.6:

The two dark green colored triangles each have an area of ½ · 3 · 7 = 10.5. The two purple colored triangles each have an area of ½ · 2 · 5 = 5. The two blue-green coloured L-shaped forms each have an area of 14.

If you sum up these areas, you get a total area of 2 · (10.5 + 5 + 14) = 59.

If you consider onlay the width of the base and the altitude of the two figures, then in both cases you get an area of ½ · 10 · 12 = 60.

The violet colored rectangular triangles have an acute angle of $\tan^{-1}(2/5) \approx 21.80°$, the dark green colored ones of $\tan^{-1}(3/7) \approx 23.20°$.

If you look closely, you can see that the "legs" of the isosceles “triangle” in the first figure are bent inwards and in the second figure outwards. If the vertices of the first figure were connected, a very narrow white band would be visible on both sides; these two bands have an area of ½ each, which results from the difference of areas: 60 – 59.

Also in the second figure an area of 59 is colored, together with the two squares left white, you get an area of 61.

✱ A 13.7:

In both figures the following four pieces are used:

The light blue colored right-angled triangle has sides with the lengths 5 and 13, thus we have an area of ½ · 5 · 13 = 32.5. The acute angle has an angular size of $\tan^{-1}(5/13) \approx 21.04°$.

The red colored right-angled triangle has sides with the lengths 8 and 21, thus we have an area of ½ · 8 · 21 = 84. The acute angle has an angular size of $\tan^{-1}(8/21) \approx 20.85°$.

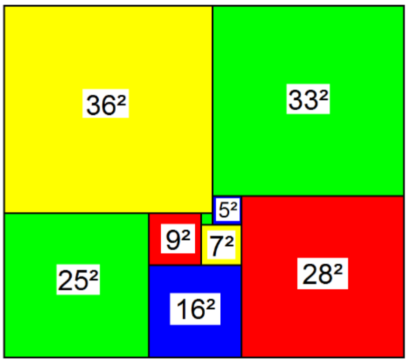
The yellow right-angled trapezoid has basic sides with the lengths 34 and 13 and an altitude with the length 8, thus we have an area of 34 · 8 – ½ · 21 · 8 = 188. The acute angle has an angular size of $\tan^{-1}(8/21) \approx 20.85°$, i.e. the red colored triangle and the yellow colored trapezium complement each other to form a rectangle.

The green colored rectangular trapezoid has base sides with the lengths 34 and 21 and an altitude with the length 8, i.e. an area of 34 · 8 - ½ · 13 · 8 = 220. The acute angle of the right-angled triangle missing in relation to a rectangle has an angular size of $\tan^{-1}(5/13) \approx 21.04°$, i.e. the blue triangle and the green trapezium complement each other to form a rectangle.

The first rectangular figure has an area of 13 · 55 = 715, the second of 21 · 34 = 714. The difference of 1 unit square is due to the diagonal boundary lines of the two trapezoids and triangles, which do not match exactly.

Chapter 14

✱ A 14.1:



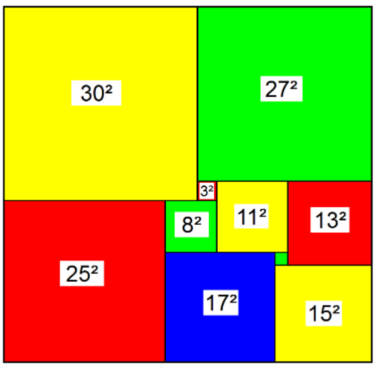
dimensions 69x61: The squares have the side lengths 2, 5, 7, 9, 16, 25, 28, 33, 36. The side lengths can be assigned to the squares when starting in the middle and going to the boundary.

Examples for horizontal cuts (from left to right):

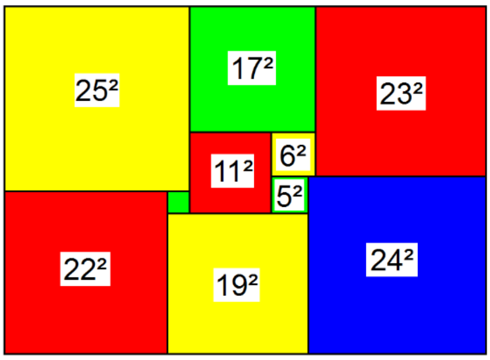
36 (yellow) + 33 (green) = 69; 25 (green) + 16 (blue) + 28 (red) = 69.

Examples for vertical cuts (from top to bottom):

36 (yellow) + 25 (green) = 61; 33 (green) + 28 (red) = 61.



dimensions 57x55



dimensions 65x47

✱ A 14.2:

c = x + 1   d = c + 1 = x + 2   e = d + 1 = x + 3   b = c + x = 2x +1

g = (e + 1) – x = 4   f = e + g = x + 7   a = f + g = x + 11

width below: a + b = 3x + 12   width above: f + e + d = 3x + 12

height on the left: a + f = 2x + 18   height on the right: b + c + d = 4x + 4

2x + 18 = 4x + 4 ⇔ x = 7

a = 18 ; b = 15 ; c = 8 ; d = 9 ; e = 10 ; f = 14 ; g = 4 ; x = 7 (dimension: 33x32)

✱ A 14.3:

(1) 33x32-rectangle from fig. 14.3: (18,5)(7,8)(14,4)(10,1)(9)

(2) 57x55-rectangle from fig. 14.5a: (30,27)(3,11,13)(25,8)(17,2)(5)

(3) 65x47-rectangle from fig. 14.5b: (25,17,23)(11,6)(5,24)(22,3)(19)

✱ A 14.4:

$h = x + 1$ $c = h + 1 = x + 2$ $d = c + 1 = x + 3$ $b = c + h = 2x + 3$

$e = x + g$ $f = e + g = x + 2g$ $a = f + g = x + 3g$

From $a + g = b + h + x$ it results: $x + 3g + g = 2x + 3 + x + 1 + x \Leftrightarrow \underline{4g = 3x + 4}$.

horizontal cut: $a + b = x + 3g + 2x + 3 = 3x + 3g + 3$ and
$f + e + d = x + 2g + x + g + x + 3 = 3x + 3g + 3$

vertical cut: $a + f = x + 3g + x + 2g = 2x + 5g$ and
$b + c + d = 2x + 3 + x + 2 + x + 3 = 4x + 8$

From $2x + 5g = 4x + 8$ it results: $\underline{5g = 2x + 8}$.

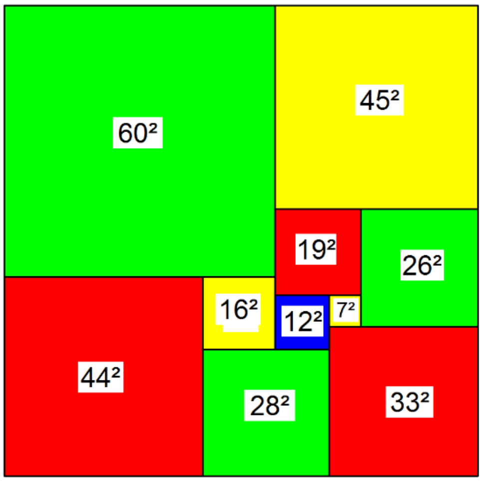
Since $5 \cdot 4g = \underline{15x + 20} = 4 \cdot 5g = \underline{8x + 32}$ it results:

$7x = 12$, i. e. $x = 12/7$ and further:
$5g = 24/7 + 8 = 80/7$, i. e. $g = 16/7$; $h = 19/7$;
$c = 26/7$; $d = 33/7$; $b = 45/7$; $e = 4$; $f = 44/7$; $a = 60/7$.

If you choose a side length of 7 length units for the smallest square, we get the following other side lengths:

$a = 60$, $b = 45$, $c = 26$, $d = 33$, $e = 28$, $f = 44$,
$g = 16$, $h = 19$ and $x = 12$, see figure on the right.

dimension: 105x104



Description by the Bouwkamp notation: (60,45)(19,26)(44,16)(12,7)(33)(28)

$f = x + h$ $g = x + f = 2x + h$ $a = g + x = 3x + h$ $e = f + h = x + h + h = x + 2h$

$d = e + 1 = x + 2h + 1$ $c = d + 1 = x + 2h + 2$ $b = c + 1 = x + 2h + 3$

Vertical cuts: $a + g = 3x + h + 2x + h = 5x + 2h$ and $b + e = x + 2h + 3 + x + 2h = 2x + 4h + 3$ and
$c + d = x + 2h + 2 + x + 2h + 1 = 2x + 4h + 3$

From this we get the condition $5x + 2h = 2x + 4h + 3 \Leftrightarrow \underline{3x = 2h + 3}$.

Horizontal cuts: $a + b + c = 3x + h + x + 2h + 3 + x + 2h + 2 = 5x + 5h + 5$ and
$g + f + e + d = 2x + h + x + h + x + 2h + x + 2h + 1 = 5x + 6h + 1$

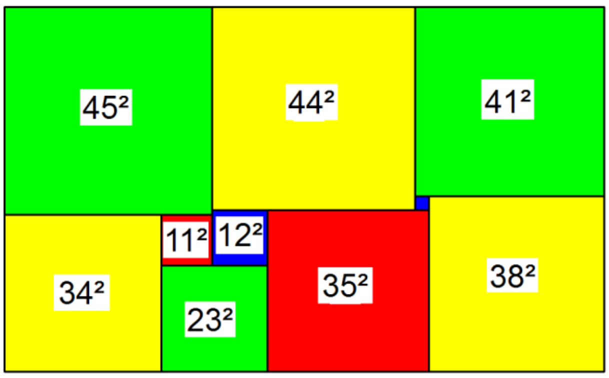
From this it results:
$5x + 5h + 5 = 5x + 6h + 1 \Leftrightarrow h = 4$.

From $3x = 2h + 3$ we get: $3x = 11$, i. e. $x = 11/3$, and further: $a = 15$, $b = 44/3$, $c = 41/3$, $d = 38/3$,
$e = 35/3$, $f = 23/3$, $g = 34/3$.

If you choose a side length of 3 length units for the smallest square, we get the following other side lengths:

$a = 45$, $b = 44$, $c = 41$, $d = 38$, $e = 35$, $f = 23$,
$g = 34$, $h = 12$ and $x = 11$, see figure on the right.

dimension: 130x79



Description by the Bouwkamp notation: (45,44,41)(3,38)(12,35)(34,11)(23) h = x + 1 g = h + x = 2x + 1 f = g + x = 3x + 1 d = f + g = 5x + 2 e = f + d = 8x + 3

c = d + g + h = 5x + 2 + 2x + 1 + x + 1 = 8x + 4

b = c + h + 1 = 8x + 4 + x + 1 + 1 = 9x + 6 a = b + 1 = 9x + 7

vertical cuts: a + e = 9x + 7 + 8x + 3 = 17x + 10 and b + c = 8x + 4 + 9x + 6 = 17x + 10

horizontal cuts: a + b = 9x + 7 + 9x + 6 = 18x + 13 and e + d + c = 8x + 3 + 5x + 2 + 8x + 4 = 21x + 9

From this it results:
18x + 13 = 21x + 9 ⇔ 3x = 4 ⇔ x = 4/3

and further: h = 7/3, g = 11/3, f = 15/3, d = 26/3,
e = 41/3, c = 44/3, b = 54/3, a = 57/3.

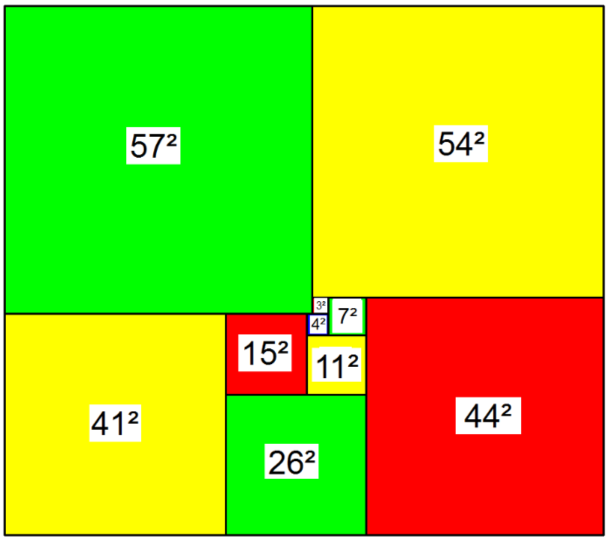
If you choose a side length of 3 length units for the smallest square, we get the following other side lengths:

a = 57, b = 54, c = 44, d = 26, e = 41, f = 15,
g = 11, h = 7 and x = 4, see figure on the right.

dimension: 111x98

Description by the Bouwkamp notation:

(57,54)(3,7,44)(41,15,4)(11)(26)



g = x + 1 e = g + 1 = x + 2 f = g + e = 2x + 3 a = f + g + x = 4x + 4

d = f – x = x + 3 c = h + d = h + x + 3 b = c + h = 2h + x + 3

a + x = b + h ⇔ 4x + 4 + x = 2h + x + 3 + h ⇔ 4x + 1 = 3h, i. e. h = 4/3 · x + 1/3

Thus we get: c = 4/3 · x + 1/3 + x + 3 = 7/3 · x + 10/3 and b = 2 · (4/3 · x + 1/3) + x + 3 = 11/3 · x + 11/3

vertical cuts: a + f = 4x + 4 + 2x + 3 = 6x + 7 and b + c = 11/3 · x + 11/3 + 7/3 · x + 10/3 = 6x + 7

horizontal cuts: a + b = 4x + 4 + 11/3 · x + 11/3 = 23/3 · x + 23/3 and
f + e + d + c = 2x + 3 + x + 2 + x + 3 + 7/3 · x + 10/3 = 19/3 · x + 34/3

From this it follows: 23/3 · x + 23/3 = 19/3 · x + 34/3 ⇔ 4/3 · x = 11/3 ⇔ x = 11/4

and further:

a = 60/4, b = 55/4, c = 39/4, d = 23/4, e = 19/4,
f = 34/4, g = 15/4, h = 16/4

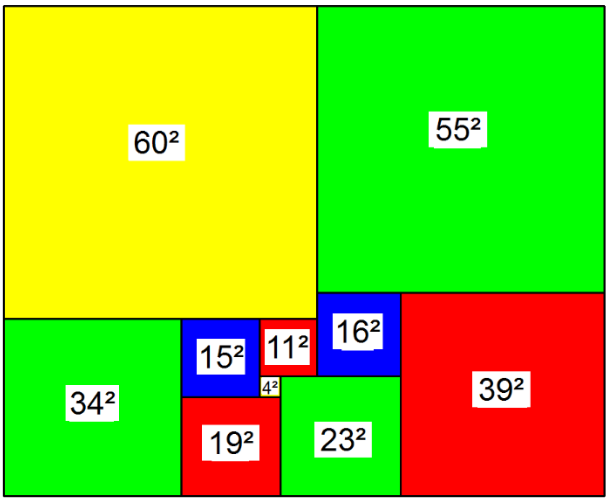
If you choose a side length of 4 length units for the smallest square, we get the following other side lengths:

a = 60, b = 55, c = 39, d = 23, e = 19, f = 34,
g = 15, h = 16 and x = 11, see figure on the right.

dimension: 115x94

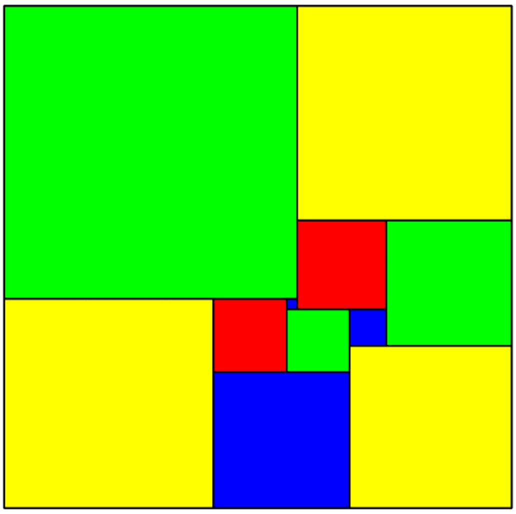
Description by the Bouwkamp notation:

(60,55)(16,39)(34,15,11)(4,23)(19)

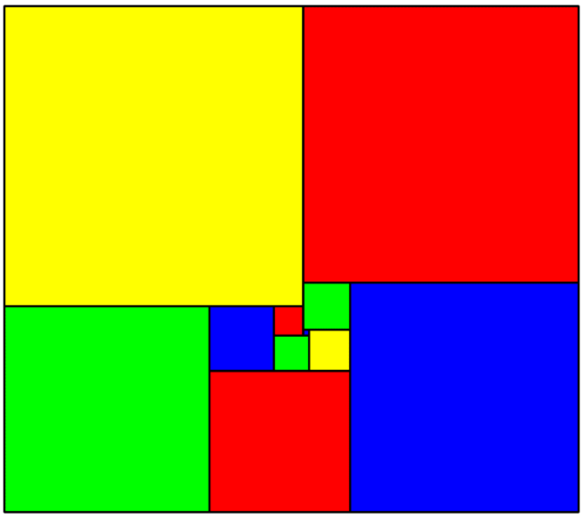


✱ A 14.5:

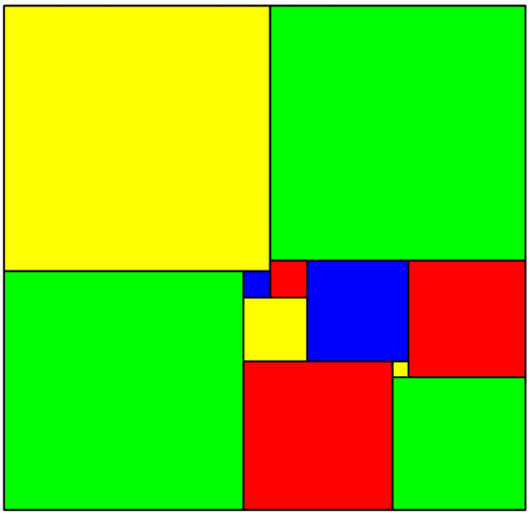
(1) (56,41)(17,24)(40,14,2)(12,7)(31)(26),



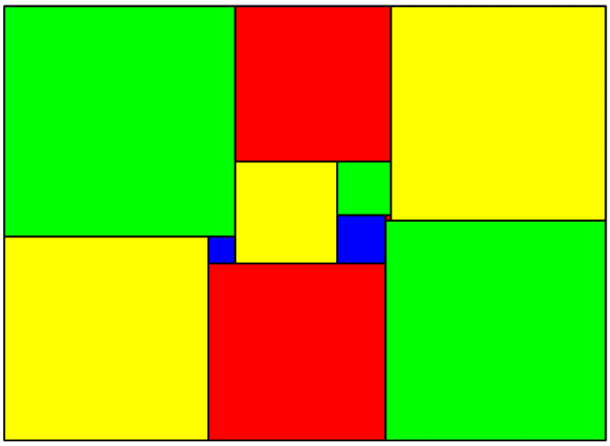
(2) (51,47)(8,39)(35,11,5)(1,7)(6)(24)



(3) (50,48)(7,19,22)(45,5)(12)(28,3)(25)



(4) (43,29,40)(19,10)(9,1)(41)(38,5)(33).



✱ A 14.6:

We can assign a side length of 1 to the small red colored square. Then we get:

$\mathbf{m} = n + 1$ $\mathbf{l} = m + 1 = n + 2$ $\mathbf{k} = l + 1 = n + 3$ $p + n = k + 1 \Leftrightarrow p + n = n + 3 + 1 \Leftrightarrow \mathbf{p} = 4$

$\mathbf{o} = n - p = n - 4$ $\mathbf{q} = o - p = n - 4 - 4 = n - 8$ $\mathbf{a} = m + n + o = n + 1 + n + n - 4 = 3n - 3$

In the lower right corner of the figure, the following relationships arise:

$\mathbf{e} = u + x$ $\mathbf{f} = e + x = u + 2x$ $\mathbf{g} = f + x = u + 3x$ $\mathbf{c} = u + e = 2u + x$

$\mathbf{h} = g - v = u + 3x - v$ $\mathbf{s} = h - v = u + 3x - 2v$

$t + u = v + g + x = v + u + 3x + x \Leftrightarrow \mathbf{t} = v + 4x$

$r + s = t + v$, also $r + u + 3x - 2v = v + 4x + v$, d. h. $\mathbf{r} = 4v - u + x$

$\mathbf{d} = r + t = 4v + x - u + v + 4x = 5v + 5x - u$

$\mathbf{b} = c + d = 2u + x + 5v + 5x - u = u + 6x + 5v$

Together with the squares considered in the beginning we have:

$h + s + q = p + k$, and further $u + 3x - v + u + 3x - 2v + n - 8 = 4 + n + 3 \Leftrightarrow \mathbf{2u - 3v + 6x = 15}$

- horizontal cuts: a + b = 3n – 3 + u + 6x + 5v

m + n + o + d + c = n + 1 + n + n – 4 + 5v + 5x – u + 2u + x = 3n + 5v + 6x + u – 3

l + k + h + g + f = n + 2 + n + 3 + u + 3x – v + u + 3x + u + 2x = 2n + 3u + 8x – v + 5

From this we get the following equation

3n + 5v + 6x + u – 3 = 2n + 3u + 8x – v + 5 ⇔ **n – 2u + 6v – 2x = 8**

- vertical cuts:

a + m + l = 3n – 3 + n + 1 + n + 2 = 5n

b + d + t + g = u + 6x + 5v + 5v + 5x – u + v + 4x + u + 3x = u + 18x + 11v

b + c + e + f = u + 6x + 5v + 2u + x + u + x + u + 2x = 5u + 10x + 5v

From this we get the following two equations

5n = u + 18x + 11v ⇔ **-5n + u + 11v + 18x = 0**

u + 18x + 11v = 5u + 10x + 5v ⇔ **-4u + 6v + 8x = 0**

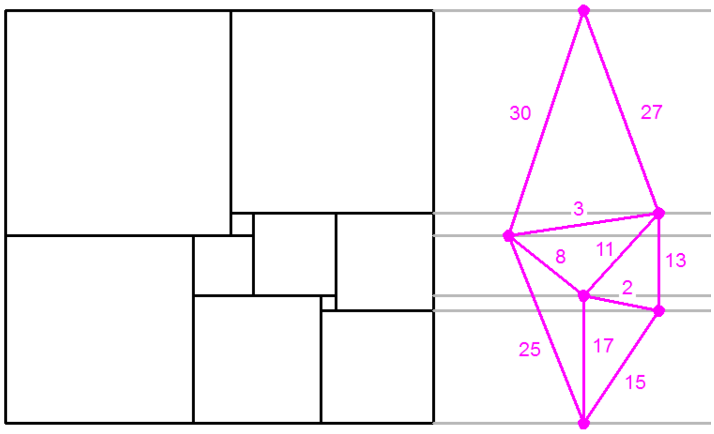
The system of equations has the following solutions: n = 11, u = 6, v = 2 und x = 1,5.

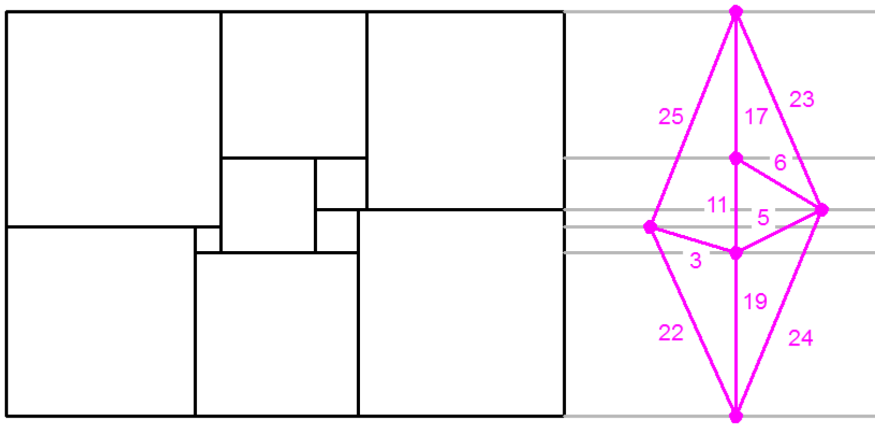
From this we get: a = 30, b = 25, c = 13,5, d = 11,5, e = 7,5, f = 9, g = 10,5, h = 8,5, k = 14, l = 13, m = 12, o = 7, p = 4, q = 3, r = 3,5, s = 6,5, t = 8.

If you choose a side length of 2 length units for the smallest square, we get the following other side lengths:

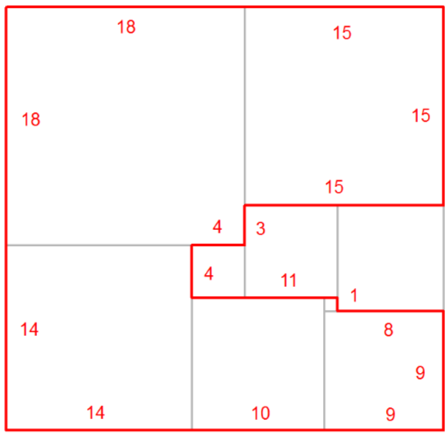
a = 60, b = 50, c = 27, d = 23, e = 15, f = 18, g = 21, h = 17, k = 28, l = 26, m = 24, n = 22, o = 14, p = 8, q = 6, r = 7, s = 13, t = 16, u = 12, v = 4 and x = 3 .

✱ A 14.7:

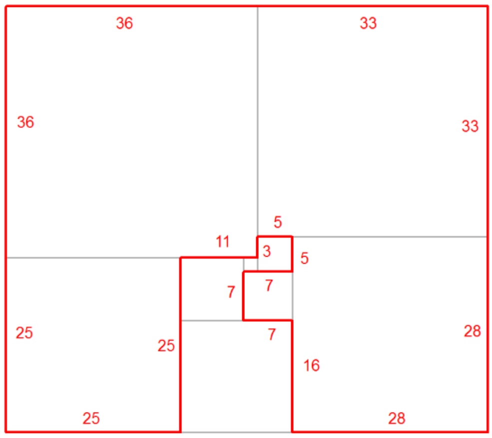




✱ A 14.8:



Total length: 168



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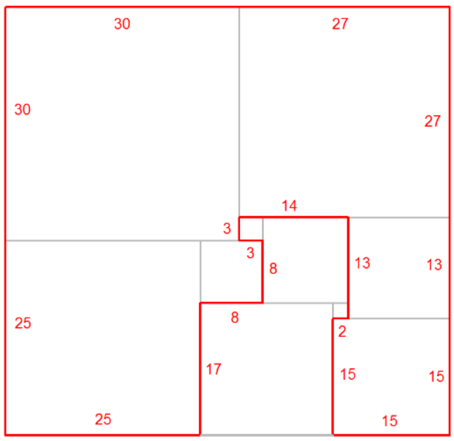
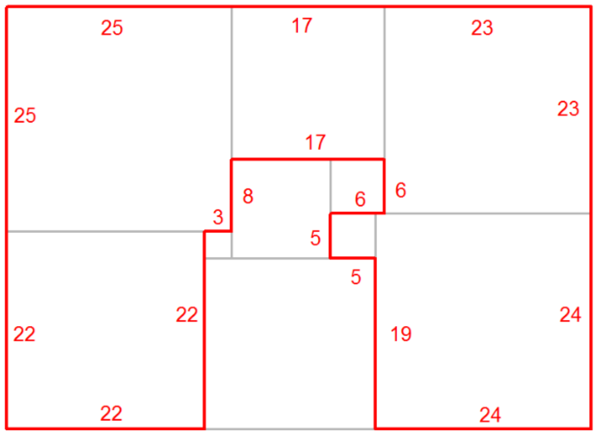


Total length: 290



Total length: 296

Chapter 15

✱ A 15.1:

Problem No.1:

r	1/10	1/9	1/8	1/7	1/6	1/5	1/4	3/10	1/3	2/5	1/2
x	9/91	8/73	7/57	6/43	5/31	4/21	3/13	21/79	2/7	6/19	1/3

Problem No.2:

r	1/10	1/9	1/8	1/7	1/6	1/5	1/4	1/3	3/7	1/2	3/5	2/3	3/4
x	36/121	8/25	28/81	3/8	20/49	4/9	12/25	1/2	12/25	4/9	3/8	8/25	12/49

✱ A 15.2:

Note that in formula 15.3 the variable x is used for the curvature of the circle to be determined, in the solution of the two problems the variable x stands for the radius of the circle. Therefore, the variable z is used here for the curvature of the circle to be determined.

- **Problem No.1**

Given is a circle with radius r and a circle with radius 1 – r as well as the outer circle with radius 1. Then the following applies for the two circles to be completed with curvature z according to formula 15.3:

$$\left[z-\left(\frac{1}{r}+\frac{1}{1-r}-1\right)\right]^2=2\cdot\left[\left(\frac{1}{r}+\frac{1}{1-r}-1\right)^2-\left(\frac{1}{r^2}+\frac{1}{(1-r)^2}+1\right)\right]$$

Auxiliary calculation:

$$\frac{1}{r}+\frac{1}{1-r}-1=\frac{1-r+r-r\cdot(1-r)}{r\cdot(1-r)}=\frac{1-r+r^2}{r\cdot(1-r)}\text{, so }\left(\frac{1}{r}+\frac{1}{1-r}-1\right)^2=\frac{(1-r+r^2)^2}{r^2\cdot(1-r)^2}$$

$$\frac{1}{r^2}+\frac{1}{(1-r)^2}+1=\frac{(1-r)^2+r^2+r^2\cdot(1-r)^2}{r^2\cdot(1-r)^2}=\frac{1-2r+r^2+r^2+r^2-2r^3+r^4}{r^2\cdot(1-r)^2}$$

$$=\frac{1-2r+3r^2-2r^3+r^4}{r^2\cdot(1-r)^2}=\frac{(1-r+r^2)^2}{r^2\cdot(1-r)^2}$$

i.e. there is zero on the right-hand side of the quadratic equation. The quadratic equation therefore only has the solution $z=\frac{1-r+r^2}{r\cdot(1-r)}$. The radius of the circles to be completed is then equal to the reciprocal of this fractional term, see solution of problem no.1.

- **Problem No.2**

Given is a circle with radius r and two circles with radius x. The outer circle is initially ignored. Then the following applies to the two circles to be supplemented with curvature z according to formula 15.3:

$$\left[z-\left(\frac{1}{r}+\frac{1}{x}+\frac{1}{x}\right)\right]^2=2\cdot\left[\left(\frac{1}{r}+\frac{1}{x}+\frac{1}{x}\right)^2-\left(\frac{1}{r^2}+\frac{1}{x^2}+\frac{1}{x^2}\right)\right]$$

Auxiliary calculation:

$$\left(\frac{1}{r}+\frac{1}{x}+\frac{1}{x}\right)^2=\left(\frac{1}{r}+\frac{2}{x}\right)^2=\frac{1}{r^2}+\frac{4}{rx}+\frac{4}{x^2}$$

$$\left(\frac{1}{r}+\frac{1}{x}+\frac{1}{x}\right)^2-\left(\frac{1}{r^2}+\frac{2}{x^2}\right)=\frac{1}{r^2}+\frac{4}{rx}+\frac{4}{x^2}-\frac{1}{r^2}-\frac{2}{x^2}=\frac{4}{rx}+\frac{2}{x^2}=\frac{2\cdot(2x+r)}{rx^2}$$

The two solutions of the quadratic equation are

$$z_1=\frac{1}{r}+\frac{2}{x}+\sqrt{\frac{4\cdot(2x+r)}{rx^2}} \text{ and } z_2=\frac{1}{r}+\frac{2}{x}-\sqrt{\frac{4\cdot(2x+r)}{rx^2}},$$

where for the negative solution z_2 applies: $z_2 = -1$, because the outer circle has the radius 1.

Therefore we have $-1=\frac{1}{r}+\frac{2}{x}-\sqrt{\frac{4\cdot(2x+r)}{rx^2}}$, i.e. $\sqrt{\frac{4\cdot(2x+r)}{rx^2}}=\frac{1}{r}+\frac{2}{x}+1=\frac{x+2r+rx}{rx}$.

Squaring the two sides of the equation gives:

$\frac{4\cdot(2x+r)}{rx^2}=\frac{(x+2r+rx)^2}{r^2x^2}$ ⇔ $4r\cdot(2x+r)=(x+2r+rx)^2$

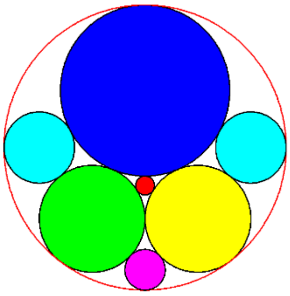
⇔ $8rx + 4r^2 = x^2 + 4r^2 + r^2x^2 + 4rx + 2rx^2 + 4r^2x$

⇔ $4rx - 4r^2x = x^2 + r^2x^2 + 2rx^2$ ⇔ $4rx \cdot (1 - r) = x^2 \cdot (1 + 2r + r^2)$ ⇔ $4r \cdot (1 - r) = x \cdot (1 + r)^2$,

see solution of problem no. 2.

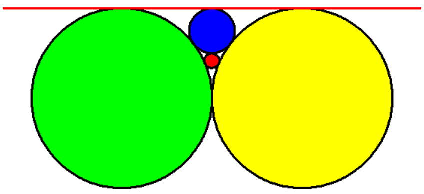
✱ A 15.3:

- **Cartesian triple (5 ; 8 ; 8)**



If we choose $r_5 = 360/3 = 120$ for the outer radius, then we have $r_1 = 360/5 = 72$ (blue); $r_2 = r_3 = 360/8 = 45$ (green, yellow), thus we get $r_4 = 360/45 = 8$ (red), and from that we have $r_6 = 360/21 = 120/7$ (pink), $r_7 = 360/12 = 30$ (light blue)

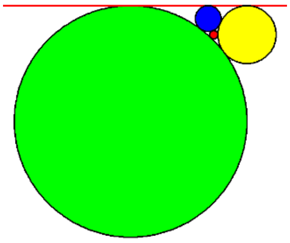
- **Cartesian triple (1 ; 1 ; 4), see chapter 15.5**



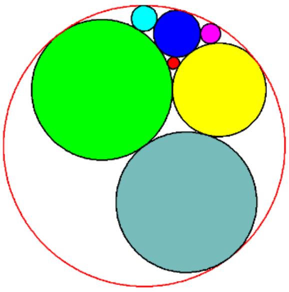
If we choose $r_1 = 12/1 = 12$ (green), $r_2 = 12/1 = 12$ (yellow) and $r_3 = 12/4 = 3$ (blue), then we get $r_4 = 12/12 = 1$ (red) and $r_5 = 12/0 = \infty$ (outer circle) and subsequently (not shown in the graphic): $r_6 = 4/3$ (which touches the yellow or green and the blue circle as well as the straight line)

- **Cartesian triple (1 ; 4 ; 9), see chapter 15.5**

If we choose r_1 = 252/1 = 252 (green), r_2 = 252/4 = 63 (yellow) and r_3 = 252/9 = 28 (blue), then we get r_4 = 252/28 = 9 (red) and r_5 = 252/0 = ∞ (outer circle), and subsequently (not shown in the graphic): r_6 = 63/4 (which touches the yellow or green and the blue circle as well as the straight line)



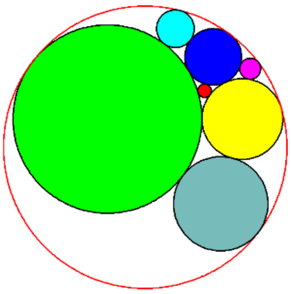
- **Cartesian triple (2 ; 3 ; 6)**



If we choose r_5 = 138/1 = 138 for the outer radius, then we get r_1 = 138/2 = 69 (green); r_2 = 138/3 = 46 (yellow) and r_3 = 128/6 = 23 (blue), and so we have r_4 = 128/23 = 6 (red), and subsequently r_6 = 138/2 = 69 (blue-grey), r_7 = 11 (turquoise), r_8 = 14 (pink).

The graph matches that of the Cartesian triple (2 ; 2 ; 3).

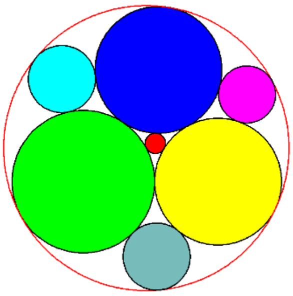
- **Cartesian triple (3 ; 7 ; 10)**



If we choose r_5 = 210/2 = 105 for the outer radius, then we get r_1 = 210/3 = 70 (green); r_2 = 210/7 = 30 (yellow) and r_3 = 210/10 = 10 (blue), and so we have r_4 = 210/42 = 5 (red), and subsequently r_6 = 210/6 = 35 (blue-grey), r_7 = 210/15 = 14 (light blue), r_8 = 210/27 = 70/9 (pink).

The graph matches that of the Cartesian triple (3 ; 6 ; 7).

- **Cartesian triple (8 ; 9 ; 9)**



If we choose r_5 = 504/4 = 126 for the outer radius, then we have r_1 = 504/8 = 63 (green); $r_2 = r_3$ = 504/9 = 56 (blue, yellow), and thus we get r_4 = 504/56 = 9 (red), and subsequently r_6 = 504/17 (blue-grey, light blue), r_7 = 504/20 = 126/5 (pink)

✱ A 15.4:

(1) The configuration of the circles is point-symmetrical and axisymmetrical:

$r_1 = r_2 = r_3 = 1$ (green, yellow, blue)

Solution of Descartes‘ equation: $x = 3 \pm 2 \cdot \sqrt{3}$, so

$r_4 = \frac{1}{3+2\cdot\sqrt{3}} \approx 0.1547$ (red) and $r_5 = -\frac{1}{3-2\cdot\sqrt{3}} \approx 2.1547$ (outer circle)

$\frac{1}{r_6} = 2\cdot\left(1+1+(3-2\cdot\sqrt{3})\right)-1 \approx 2.072$, i.e. $r_6 \approx$ 0.,483 (light blue).

(2) $r_1 = r_2 = 5$ (green, yellow), $r_3 = 8$ (blue)

Solution of Descartes‘ equation:

$$x = \left(\frac{1}{5}+\frac{1}{5}+\frac{1}{8}\right) \pm 2\cdot\sqrt{\frac{1}{5}\cdot\frac{1}{5}+\frac{1}{5}\cdot\frac{1}{8}+\frac{1}{5}\cdot\frac{1}{8}} = \frac{21}{40} \pm 2\cdot\sqrt{\frac{9}{100}} = \frac{21}{40} \pm \frac{24}{40}$$

From this we get $r_4 = \frac{8}{9} \approx 0.889$ (red) and $r_5 = \frac{40}{3} \approx 13.333$ (outer circle).

wanted: two circles, which touch the following three circles	radius already known	$x_1 = 2\cdot(k_1+k_2+k_3)-x_2$	radius of the new circle
green (5), blue (8), outer circle (40/3)	yellow (5)	$2\cdot\left(\frac{1}{5}+\frac{1}{8}-\frac{3}{40}\right)-\frac{1}{5}=\frac{3}{10}$	$\frac{10}{3} \approx 3.3$ (light blue)
green (5), yellow (5), outer circle (40/3)	blue (8)	$2\cdot\left(\frac{1}{5}+\frac{1}{5}-\frac{3}{40}\right)-\frac{1}{8}=\frac{21}{40}$	$\frac{40}{21} \approx 1.905$ (pink)

(3) $r_1 = 6$ (green), $r_2 = 5$ (yellow), $r_3 = 3$ (blue)

$$2\cdot\left(\frac{1}{6^2}+\frac{1}{5^2}+\frac{1}{3^2}+\frac{1}{x^2}\right) = \left(\frac{1}{6}+\frac{1}{5}+\frac{1}{3}+\frac{1}{x}\right)^2 \Leftrightarrow 2\cdot\left(\frac{161}{900}+\frac{1}{x^2}\right) = \left(\frac{7}{10}+\frac{1}{x}\right)^2 \Leftrightarrow$$

$$\frac{1}{x^2}-\frac{7}{5x}-\frac{119}{900}=0 \Leftrightarrow x^2+\frac{180}{17}x=\frac{900}{119} \Leftrightarrow x=-\frac{90}{17}\pm\frac{\sqrt{504000}}{119},$$

therefore $r_4 \approx 0.672$ and $r_5 \approx 11.260$

wanted: two circles, which touch the following three circles	radius already known	$x_1 = 2\cdot(k_1+k_2+k_3)-x_2$	radius of the new circle
green (6), blue (3), outer circle(11.260)	yellow (5)	$2\cdot\left(\frac{1}{6}+\frac{1}{3}-\frac{1}{11.260}\right)-\frac{1}{5}\approx 0.622$	1.607 (light blue)
blue (3), yellow (5), outer circle (11.260)	green (6)	$2\cdot\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{11.260}\right)-\frac{1}{6}\approx 0.722$	1.384 (pink)

green (6), yellow (5), outer circle (11.260)	blue (3)	$2 \cdot \left(\frac{1}{6}+\frac{1}{5}-\frac{1}{11.260}\right)-\frac{1}{3} \approx 0.222$	4.497 (blue-grey)

✱ A 15.5:

(1) From the curvatures of the touching circles $k_1 = 14$, $k_2 = 15$ and $k_3 = 11$ results according to formula 15.5: $(14+15+11) \pm 2 \cdot \sqrt{14 \cdot 15 + 14 \cdot 11 + 15 \cdot 11} = 40 \pm 2 \cdot \sqrt{529} = 40 \pm 46$, thus $k_4 = -6$ (curvature of the outer circle) and $k_5 = 86$.

From this, the curvatures of the neighbouring circles are obtained with the help of formula 15.4:

2 · (14 + 15 – 6) – 11 = 35 (touching the circles with the curvatures k_1, k_2 und k_4),

2 · (14 + 11 – 6) – 15 = 23 (touching the circles with the curvatures k_1, k_3 und k_4) etc.

(2) From the curvatures of the touching circles $k_1 = 8$, $k_2 = 8$ und $k_3 = 5$ results according to formula 15.5: $(8+8+5) \pm 2 \cdot \sqrt{8 \cdot 8 + 8 \cdot 5 + 8 \cdot 5} = 21 \pm 2 \cdot \sqrt{144} = 21 \pm 24$, thus $k_4 = -3$ (curvature of the outer circle) and $k_5 = 45$.

From this, the curvatures of the neighbouring circles are obtained with the help of formula 15.4:

2 · (8 + 8 – 3) – 5 = 21 (touching the circles with the curvatures k_1, k_2 und k_4),

2 · (8 + 5 – 3) – 8 = 12 (touching the circles with the curvatures k_1, k_3 und k_4) etc.

(3) From the curvatures of the touching circles $k_1 = 4$, $k_2 = 12$ und $k_3 = 13$ results according to formula 15.5: $(4+12+13) \pm 2 \cdot \sqrt{4 \cdot 12 + 4 \cdot 13 + 12 \cdot 13} = 29 \pm 2 \cdot \sqrt{256} = 29 \pm 32$, thus $k_4 = -3$ (curvature of the outer circle) and $k_5 = 61$.

From this, the curvatures of the neighbouring circles are obtained with the help of formula 15.4:

2 · (12 + 13 – 3) – 4 = 40 (touching the circles with the curvatures k_2, k_3 und k_4),

2 · (13 + 4 – 3) – 12 = 16 (touching the circles with the curvatures k_1, k_3 und k_4) etc.

✱ A 15.6:

In example 2 we have $r = \frac{1}{2}$ and $r_0 = \frac{1}{3}$. From this, the curvatures of the neighbouring circles K_1 are $\frac{1}{r_1} = \frac{1}{\frac{1}{3}} + \frac{1}{\frac{1}{6}} - \frac{1}{\frac{1}{2}} = 7$, and consequently:

$$\frac{1}{r_2} = \frac{4}{\frac{1}{3}} + \frac{1}{\frac{1}{6}} - \frac{4}{\frac{1}{2}} = 10, \quad \frac{1}{r_3} = \frac{9}{\frac{1}{3}} + \frac{1}{\frac{1}{6}} - \frac{9}{\frac{1}{2}} = 15 \text{ and } \frac{1}{r_4} = \frac{16}{\frac{1}{3}} + \frac{1}{\frac{1}{6}} - \frac{16}{\frac{1}{2}} = 22 .$$

Generally: $\frac{1}{r_{n+1}} = \frac{(n+1)^2}{\frac{1}{3}} + \frac{1}{\frac{1}{6}} - \frac{(n+1)^2}{\frac{1}{2}} = (n+1)^2 + 6 = n^2 + 2n + 7$

and thus the sequence of curvatures is 6, 7, 10, 15, 22, 31, 42, …

✱ A 15.7:

Example: 12, 13, 16, 21, …

Here we have $r = 1/3$ (outer circle) and $r_0 = 1/4$

$$\frac{1}{r_1} = \frac{1}{\frac{1}{4}} + \frac{1}{\frac{1}{12}} - \frac{1}{\frac{1}{3}} = 13; \quad \frac{1}{r_2} = \frac{4}{\frac{1}{4}} + \frac{1}{\frac{1}{12}} - \frac{4}{\frac{1}{3}} = 16; \quad \frac{1}{r_3} = \frac{9}{\frac{1}{4}} + \frac{1}{\frac{1}{12}} - \frac{9}{\frac{1}{3}} = 21; \quad \frac{1}{r_4} = \frac{16}{\frac{1}{4}} + \frac{1}{\frac{1}{12}} - \frac{16}{\frac{1}{3}} = 28$$

Generally: $\frac{1}{r_{n+1}} = \frac{(n+1)^2}{\frac{1}{4}} + \frac{1}{\frac{1}{12}} - \frac{(n+1)^2}{\frac{1}{3}} = (n+1)^2 + 12 = n^2 + 2n + 13$

✱ A 15.8:

According to formula 15.4, the curvature k_4 of a circle, which touches the circles with the curvatures

$k_1 = n^2$, $k_2 = (n + k)^2$, $k_3 = (2n + k)^2$:

$k_4 = 2 \cdot (n^2 + (n + k)^2 + (2n + k)^2) - 0 = 2 \cdot (n^2 + n^2 + 2nk + k^2 + 4n^2 + 4nk + k^2)$, thus

$k_4 = 4 \cdot (3n^2 + 3nk + k^2)$ with $n, k \in \mathbb{N}$

Examples: For n = 1 and k = 0 we get: $k_4 = 4 \cdot (3 + 0 + 0) = 12$, see fig. 15.7a; for n = 1 and k = 1 we get: $k_4 = 4 \cdot (3 + 3 + 1) = 28$, see fig. 15.7b.

✱ A 15.9:

From $k_1{}^2 + k_2{}^2 + k_3{}^2 = 2 \cdot (k_1 k_2 + k_1 k_3 + k_2 k_3)$ we get

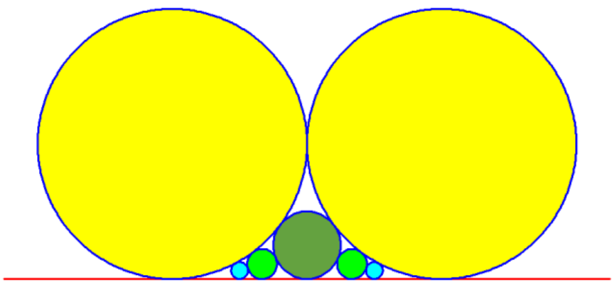
$k_3^2 - 2 \cdot k_3 \cdot (k_1 + k_2) = 2 \cdot k_1 \cdot k_2 - k_1^2 - k_2^2 \Leftrightarrow (k_3 - (k_1 + k_2))^2 = 2 \cdot k_1 \cdot k_2 - k_1^2 - k_2^2 + (k_1 + k_2)^2 \Leftrightarrow$

$(k_3 - (k_1 + k_2))^2 = 4 \cdot k_1 \cdot k_2$, and therefore $k_3 = (k_1 + k_2) \pm 2 \cdot \sqrt{k_1 k_2}$.

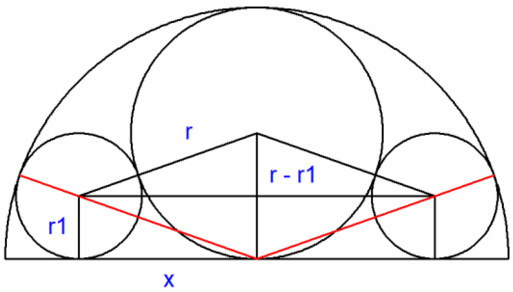
In the example with $k_1 = k_2 = 1$, the two solutions are $k_3 = 0$ respectively $k_3 = 4$.

In the example with $k_1 = 1$ (yellow) and $k_2 = 4$ (green), the two solutions are $k_3 = 1$ respectively $k_3 = 9$, i.e. apart from the circle with radius $r_3 = 1/9$ (light blue), there is another circle with radius $r_3 = 1$.

etc.



✱ A 15.10:

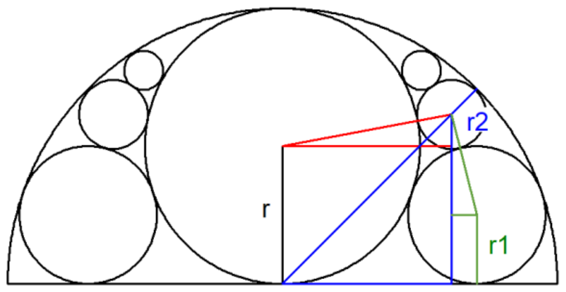


For the right-angled triangle determined by the centres of the yellow-coloured and the green-coloured circle, the following applies:

$x^2 = (r + r_1)^2 - (r - r_1)^2$, and from this we get $x^2 = 4 \cdot r \cdot r_1$, therefore with $r = \frac{1}{2}$: $x^2 = 2 \cdot r_1$.

For the distance x also applies: $x^2 = (1 - r_1)^2 - r_1{}^2 = 1 - 2r_1$

From this we get: $1 - 2r_1 = 2r_1 \Leftrightarrow r_1 = \frac{1}{4}$, i.e. $x = \frac{1}{2} \cdot \sqrt{2}$.



To determine the centre M_2 ($x_2|y_2$) and the radius r_2, consider the following triangles:

red: $(y_2 - \frac{1}{2})^2 + x_2^2 = (\frac{1}{2} + r_2)^2$

green: $(y_2 - \frac{1}{4})^2 + (\frac{1}{2} \cdot \sqrt{2} - x_2)^2 = (\frac{1}{4} + r_2)^2$

blue: $x_2^2 + y_2^2 = (1 - r_2)^2$

Transforming these three equations gives

$$x_2^2 + y_2^2 - r_2^2 = y_2 + r_2$$

$$x_2^2 + y_2^2 - r_2^2 = \sqrt{2} \cdot x_2 + \frac{1}{2} y_2 + \frac{1}{2} r_2 - \frac{1}{2}$$

$$x_2^2 + y_2^2 - r_2^2 = 1 - 2r_2$$

Therefore we have the two equations $y_2 + r_2 = 1 - 2r_2$, i.e. $y_2 = 1 - 3r_2$, and

$y_2 + r_2 = \sqrt{2} \cdot x_2 + \frac{1}{2} y_2 + \frac{1}{2} r_2 - \frac{1}{2}$, i.e. $\sqrt{2} \cdot x_2 - \frac{1}{2} y_2 - \frac{1}{2} r_2 = \frac{1}{2}$.

If one replaces the variable y_2, one obtains from this $\sqrt{2} \cdot x_2 + r_2 = 1$, i.e. $x_2 = \frac{\sqrt{2}}{2} \cdot (1 - r_2)$.

For example, if one substitutes this into the third equation $x_2^2 + y_2^2 = (1 - r_2)^2$, we get:

$\frac{1}{2} \cdot (1 - r_2)^2 + (1 - 3r_2)^2 = (1 - r_2)^2$, and

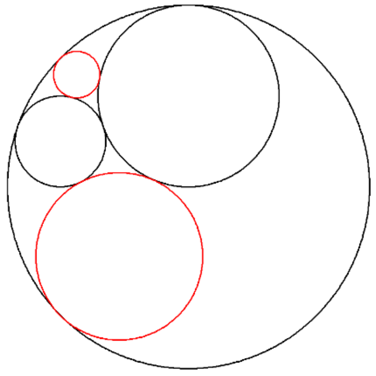
$\frac{1}{2} - r_2 + \frac{1}{2} \cdot r_2^2 + 1 - 6r_2 + 9r_2^2 = r_2^2 - 2r_2 + 1 \Leftrightarrow \frac{17}{2} r_2^2 - 5r_2 = -\frac{1}{2} \Leftrightarrow$

$r_2^2 - \frac{10}{17} r_2 = -\frac{1}{17} \Leftrightarrow (r_2 - \frac{5}{17})^2 = \frac{25}{289} - \frac{1}{17} (= \frac{8}{289}) \Leftrightarrow r_2 \approx 0.128$ (which is the smaller of the two solutions)

From this we get: $x_2 \approx 0.617$ und $y_2 \approx 0.617$.

The radius can also be calculated using Descartes' formula (15.3) - ignoring the restriction to a semicircle. Replacing $k_1 = \frac{1}{0,5} = 2$, $k_2 = \frac{1}{0,25} = 4$ and $k_3 = \frac{1}{-1} = -1$ leads to $(k_4 - 5)^2 = \pm 2\sqrt{3}$, and therefore $r_4 = \frac{1}{5 \pm 2\sqrt{3}}$.

The smaller of the two solutions is the radius calculated above; the following figure also shows the other touching circle.



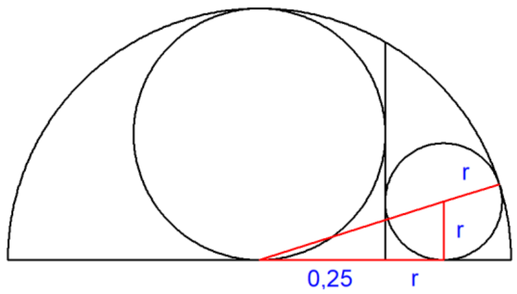
The radius of the circle coloured red in the problem can then be calculated with the help of Vieta's theorem (formula 15.4): The radii of the yellow circle (0.5) and blue circle (0.128) as well as the radius of the outer circle (1) are known; a solution is then already known, namely the radius of the green circle (0.25):

$2\cdot\left(\frac{1}{0.5}+\frac{1}{0.128}-\frac{1}{1}\right)-\frac{1}{0.25}\approx 13.657\approx\frac{1}{0.073}$. Therefore the radius is 0.073!

✱ A 15.11:

left figure: The following applies to the diagonal of the square: $\sqrt{2}=2r\cdot(1+\sqrt{2})$ and

$$r=\frac{\sqrt{2}}{2\cdot(1+\sqrt{2})}=\frac{\sqrt{2}\cdot(\sqrt{2}-1)}{2}=1-\tfrac{1}{2}\cdot\sqrt{2}\approx 0.293.$$



right figure: In the rectangular triangle whose hypotenuse is equal to the line segment between the centre of the semicircle and the centre of the small circle on the right we have:

$(\frac{1}{4}+r)^2+r^2=(\frac{1}{2}-r)^2$, therefore $r^2+\frac{3}{2}r=\frac{3}{16}\;\Leftrightarrow\;(r+\frac{3}{4})^2=\frac{3}{4}\;\Leftrightarrow\;r=\frac{\sqrt{3}}{2}-\frac{3}{4}\approx 0.116$.

✱ A 15.12:

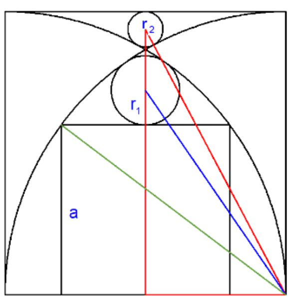
The following applies to the radius R = 1 of the quarter circle (= side length of the square):

$1 = r_1 + r_1\cdot\sqrt{2}$, also $r_1=\frac{1}{1+\sqrt{2}}=\frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}=\sqrt{2}-1\approx 0.414$

and further for the diagonal of the square $\sqrt{2} = 1 + (r_2 + r_2\cdot\sqrt{2})$, therefore

$$r_2=\frac{\sqrt{2}-1}{1+\sqrt{2}}=\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}=(\sqrt{2}-1)^2=3-2\cdot\sqrt{2}\approx 0.172$$

✱ A 15.13:



right-angled triangle with red hypotenuse: $(1-r_2)^2+(\frac{1}{2})^2=(1+r_2)^2$, therefore $4r_2=\frac{1}{4}$, i.e. $r_2=\frac{1}{16}$

right-angled triangle with green hypotenuse: $(\frac{1}{2}+\frac{a}{2})^2+a^2=1^2$, also $\frac{5}{4}a^2+\frac{1}{2}a=\frac{3}{4}\;\Leftrightarrow\;a^2+\frac{2}{5}a=\frac{3}{5}$

$\Leftrightarrow\;(a+\frac{1}{5})^2=\frac{16}{25}\;\Leftrightarrow\;a=\frac{3}{5}$

right-angled triangle with blue hypotenuse: $(a+r_1)^2+(\frac{1}{2})^2=(1-r_1)^2$, and with $a=\frac{3}{5}$ we get:

$\frac{16}{5}r_2=\frac{39}{100} \Leftrightarrow r_2=\frac{39}{320}$

✱ A 15.14:

Between the square's diagonal (length $\sqrt{2}$) and the radius *r* of the red coloured circle the following relation applies: $2r_1+2\cdot(1-2r_1)=\sqrt{2}$, i.e. $2r_1=2-\sqrt{2} \Leftrightarrow r_1=1-\frac{1}{2}\cdot\sqrt{2}\approx 0.293$

If you then draw a line between the left upper vertex of the square to the centre of a small circle (here then top right), then this is the hypotenuse in a right-angled triangle with legs of the lengths *x* and 1 – *x* and a hypotenuse of length 1 – r_2. Therefore we have

$x^2+(1-x)^2=(1-r_2)^2 \Leftrightarrow 2x^2-2x+1=(1-r_2)^2$

For the diagonal of the larger square we have: $2\cdot r_1+2\cdot r_2+2\cdot x\cdot\sqrt{2}=\sqrt{2}$, thus

$r_2=\frac{1}{2}\sqrt{2}-(1-\frac{1}{2}\cdot\sqrt{2})-x\cdot\sqrt{2}=(\sqrt{2}-1)-x\cdot\sqrt{2}$ and therefore

$1-r_2=x\cdot\sqrt{2}+(2-\sqrt{2})$, further applies

$(1-r_2)^2=2x^2+2\cdot(2-\sqrt{2})\cdot\sqrt{2}\cdot x+(6-4\cdot\sqrt{2})=2x^2+(4\cdot\sqrt{2}-4)\cdot x+(6-4\cdot\sqrt{2})$

With the term standing on the right and the term from above we get:

$2x^2-2x+1=2x^2+(4\cdot\sqrt{2}-4)\cdot x+(6-4\cdot\sqrt{2}) \Leftrightarrow (4\cdot\sqrt{2}-2)\cdot x=4\cdot\sqrt{2}-5$, i.e.

$x=\frac{4\cdot\sqrt{2}-5}{4\cdot\sqrt{2}-2}=\frac{1}{14}\cdot(11-6\cdot\sqrt{2})\approx 0.180$, and therefore

$r_2=(\sqrt{2}-1)-\frac{1}{14}\cdot(11-6\cdot\sqrt{2})\cdot\sqrt{2}=\frac{3}{14}\cdot\sqrt{2}-\frac{1}{7}\approx 0.160$

✱ A 15.15:

right-angeld triangle OAM_1 (red): $(\frac{1}{2})^2+r_1^2=(1-r_1)^2$, therefore $\frac{1}{4}+r_1^2=1-2r_1+r_1^2 \Leftrightarrow 2r_1=\frac{3}{4} \Leftrightarrow r_1=\frac{3}{8}$

Other right-angled triangles (blue):

OBM_2: $OB^2+BM_2^2=OM_2^2 \Leftrightarrow (1-r_2)^2+BM_2^2=(1+r_2)^2$

BCM_2: $BC^2+BM_2^2=CM_2^2 \Leftrightarrow r_2^2+BM_2^2=(1-r_2)^2$

Dissolve the two equations to BM_2^2 and equate:

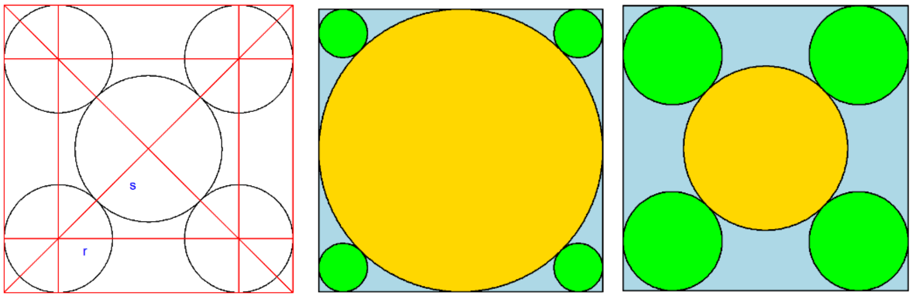
$BM_2^2=(1+r_2)^2-(1-r_2)^2=(1-r_2)^2-r_2^2$ and then

$1+2r_2+r_2^2-1+2r_2-r_2^2=1-2r_2+r_2^2-r_2^2 \Leftrightarrow 6r_2=1 \Leftrightarrow r_2=\frac{1}{6}$

right-angled triangle OAM_3 (green): $(\frac{1}{2})^2+(1-r_3)^2=(1+r_3)^2$, therefore

$\frac{1}{4}+1-2r_3+r_3^2=1+2r_3+r_3^2 \Leftrightarrow 4r_3=\frac{1}{4} \Leftrightarrow r_3=\frac{1}{16}$

✱ A 15.16:



Regardless of which of the positions of the circles one looks at, the following applies: The 5 circles have an area of $A = \pi \cdot (s^2 + 4r^2)$.

The diagonal of the square has the length $\sqrt{2}$; it is composed of twice the length of the radius r and twice the length of the radius s, and two half diagonals in each of the smaller squares in the corners; these distances have the length $r \cdot \sqrt{2}$. Altogether, therefore, the following applies $2 \cdot r \cdot \sqrt{2} + 2 \cdot r + 2 \cdot s = \sqrt{2}$:

From this condition one can derive a term for s: $s = \frac{1}{2} \cdot \sqrt{2} - (\sqrt{2}+1) \cdot r$ and after substitution one then obtains a quadratic function

$$A(r) = \pi \cdot (s^2 + 4r^2) = \pi \cdot \left(\tfrac{1}{2} - (2+\sqrt{2}) \cdot r + (7 + 2 \cdot \sqrt{2}) \cdot r^2\right),$$

whose graph represents a parabola opened upwards. The largest values are assumed to be on the right or left edge of the domain of the problem.

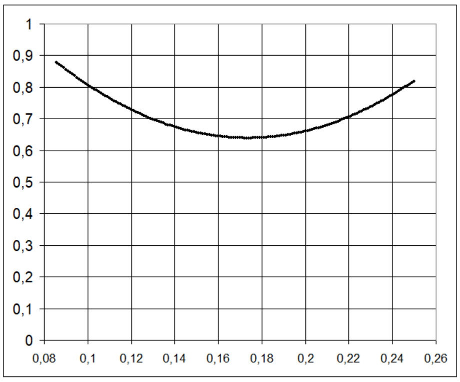
Therefore, we determine this domain:

(1) The circle with radius *s* touches the sides of the square, i.e. *s* = 0,5. In this case we have

$$2 \cdot (\sqrt{2}+1) \cdot r + 2 \cdot 0{,}5 = \sqrt{2}\text{, therefore } r = \frac{\sqrt{2}-1}{2 \cdot (\sqrt{2}+1)} = \frac{(\sqrt{2}-1)(\sqrt{2}-1)}{2 \cdot (\sqrt{2}+1)(\sqrt{2}-1)} = \frac{3-2\sqrt{2}}{2} \approx 0.086$$

(2) The vier circles in the corners touch each other, i.e. *r* = 0,25, therefore the domain is: $0.086 \le r \le 0.25$

The function values are: A(0.086) ≈ 0.878 and A(2.5) ≈ 0.819, i.e. the maximum is assumed in the case that the inner circle touches the square sides.



What remains to be calculated:

– Which area is covered when the five radii are equal?
  Then applies: $(2 \cdot \sqrt{2} + 4) \cdot r = \sqrt{2}$ and therefore
  $r = \frac{1}{2} \cdot (\sqrt{2}-1) \approx 0{,}207$ with A ≈ 0.674.

– Which is the minimum of coverage?
  From the graph you can see that this minimum is r ≈ 0.174.
  The exact value can be found by determining the vertex of the quadratic parabola. Therefore we get: $r = \frac{10 + 3 \cdot \sqrt{2}}{82} \approx 0.1737$

✱ A 15.17:

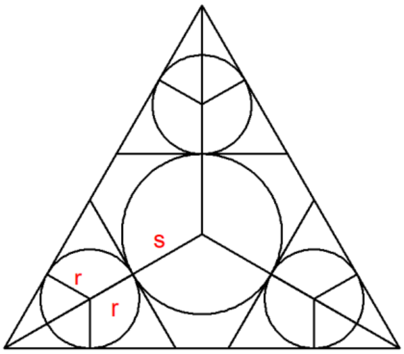
Medium figure: The large circle is the incircle of the equilateral triangle with side length 1, i.e. its radius is one third as large as its height, so $s = \frac{1}{6} \cdot \sqrt{3}$. Tangents through the points of contact separate smaller equilateral triangles from the large equilateral triangle. For the circles at the corners it applies that they are incircles of these "remaining" equilateral triangles, i.e. they have a radius r that is one third as large as the height of these equilateral partial triangles, thus: $r = \frac{1}{3} \cdot \frac{1}{6} \cdot \sqrt{3} = \frac{1}{18} \cdot \sqrt{3} \approx 0.096$.

So we get for the area: $A = \pi \cdot (\frac{1}{36} \cdot 3 + 3 \cdot \frac{1}{324} \cdot 3) = \pi \cdot \frac{1}{9}$, that is 80,6% oft he area $A_{Triangle} = \frac{1}{4} \cdot \sqrt{3}$.

Right figure: Here all circles have the same size; they are incircles of the 4 equilateral triangles of the same size, which are obtained by drawing the tangents through the points of contact. The following therefore applies here: $r = \frac{1}{12} \cdot \sqrt{3}$, i.e. $A = \frac{1}{12} \cdot \pi$; the covered portion therefore is 60.5%.

Left figure: If you connect the centres of the three large circles with each other, you get an equilateral triangle with side length 2r. If you then drop the perpendiculars from these centres onto the sides of the triangle, you see: The height of the initial triangle is as great as the height of the inner triangle of the centre points plus r (below) and plus 2r (above), thus: $\frac{1}{2} \cdot \sqrt{3} = r + \frac{2r}{2} \cdot \sqrt{3} + 2r$. Resolved to r this is: $r = \frac{\sqrt{3}-1}{4}$.

Before one now painstakingly determines the radius of the inner circle, one should rather consider the geometric situation in general:



One can always regard the corner circles with radius r as incircles of equilateral triangles, which are created by the tangents through the points of contact with the centre circle.

Then the following applies to the height h of these triangles in the corners: h = 3r .

The distance of the centre of the initial triangle from the three vertices of the triangle (radius of the circumcircle) is

$h + s = \frac{2}{3} \cdot (\frac{1}{2} \cdot \sqrt{3}) = \frac{1}{3} \cdot \sqrt{3}$, therefore $s = \frac{1}{3} \cdot \sqrt{3} - 3r$

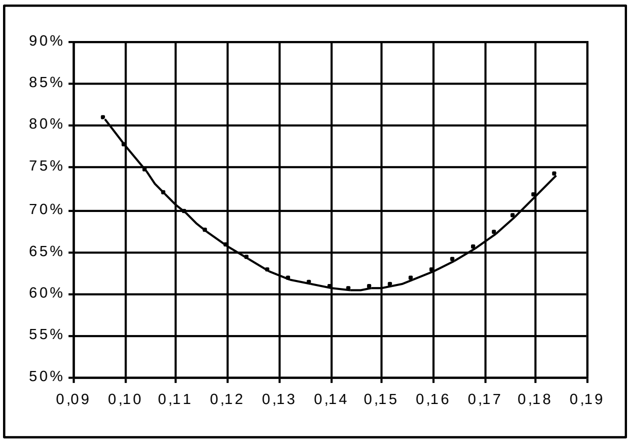
Thus the following applies to the covered part of the equilateral triangle:

$A = \pi \cdot (3r^2 + s^2) = \pi \cdot (3r^2 + (\frac{1}{3} \cdot \sqrt{3} - 3r)^2) = \pi \cdot (3r^2 + \frac{1}{3} - 2 \cdot \sqrt{3} \cdot r + 9r^2) = \pi \cdot (12r^2 + \frac{1}{3} - 2 \cdot \sqrt{3} \cdot r)$.

This is the term of a quadratic function with the variable r: $A = 12\pi \cdot (r^2 - \frac{1}{6} \cdot \sqrt{3} \cdot r + \frac{1}{36})$

The minimum of the function is at 0.144, i.e. for the case where four equally sized circles are drawn. The maximum of the function must lie at one of the two boundary values of the definition set. This domain is given by the interval $\frac{1}{18} \cdot \sqrt{3} \approx 0.096 \le r \le \frac{1}{4} \cdot (\sqrt{3} - 1) \approx 0.183$.

The following illustration shows that the maximum coverage exists if the inner circle of the equilateral triangle is drawn first and then the smaller circles are drawn in each of the corners.



✱ A 15.18:

If we denote the sides of the triangle (as usual) with a (short cathetus), b (long cathetus) and c (= 1), then the following applies: $a^2 + b^2 = 1$, the following also applies to the area: $(2r)^2 + 4 \cdot \frac{1}{2} \cdot a \cdot b = 1$. Further applies: $b = a + 2r$, so we have $(2r)^2 = (b-a)^2 = b^2 - 2ab + a^2$. But these equations are not independent, as you can see if you insert into the equation for the area: $b^2 - 2ab + a^2 + 2ab = 1$.

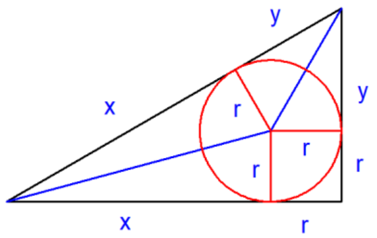
The formula for the radius of an incircle in a right-angled triangle helps further here: $r = \frac{1}{2} \cdot (a+b-c)$.

Insert in $b = a + 2r$ we get: $b = a + (a+b-1)$, i.e. 2*a* = 1, so we see, that the triangles are half equilateral triangles. Therefore it applies $b = \frac{1}{2} \cdot \sqrt{3}$ and therefore

$$r = \tfrac{1}{2} \cdot (\tfrac{1}{2} + \tfrac{1}{2} \cdot \sqrt{3} - 1) = \tfrac{1}{2} \cdot (\tfrac{1}{2} \cdot \sqrt{3} - \tfrac{1}{2}) = \tfrac{1}{4} \cdot (\sqrt{3} - 1) \approx 0.183 .$$

Note: For the area A of a right-angled triangle with an incircle radius r, the following applies

$$A = 2 \cdot \tfrac{1}{2} \cdot x \cdot r + 2 \cdot \tfrac{1}{2} \cdot y \cdot r + r^2 = x \cdot r + y \cdot r + r^2 .$$



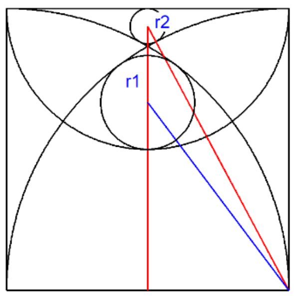
On the one hand side we have

$$A = (x \cdot r + r^2) + (y \cdot r + r^2) - r^2 = (x+r) \cdot r + (y+r) \cdot r - r^2$$
$$= a \cdot r + b \cdot r - r^2$$

on the other hand we have $A = (x+y) \cdot r + r^2 = c \cdot r + r^2$,

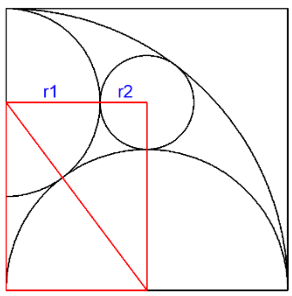
so we get: $a \cdot r + b \cdot r - r^2 = c \cdot r + r^2$, i.e. $a + b - r = c + r \Leftrightarrow a + b - c = 2r$, and therefore $r = \frac{1}{2} \cdot (a+b-c)$

✱ A 15.19:



rectangular triangle with blue hypotenuse $(\frac{1}{2} + r_1)^2 + (\frac{1}{2})^2 = (1 - r_1)^2$, thus $3r_1 = \frac{1}{2}$ and so $r_1 = \frac{1}{6}$.

rectangular triangle with red hypoteuse: $(1 - r_2)^2 + (\frac{1}{2})^2 = (1 + r_2)^2$, thus $4r_2 = \frac{1}{4}$ and so $r_2 = \frac{1}{16}$

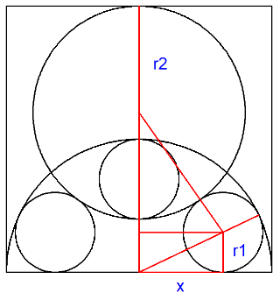


For both rectangular triangles applies:

$(1 - r_1)^2 + (\frac{1}{2})^2 = (\frac{1}{2} + r_1)^2$, thus $3r_1 = 1$ and therefore $r_1 = \frac{1}{3}$.

$(r_1 + r_2)^2 + (\frac{1}{2} + r_2)^2 = (\frac{1}{2} + r_1)^2$, thus $(\frac{1}{3} + r_2)^2 + (\frac{1}{2} + r_2)^2 = (\frac{1}{2} + \frac{1}{3})^2$; from this we get:

$$\tfrac{1}{9} + \tfrac{2}{3} r_2 + r_2^2 + \tfrac{1}{4} + r_2 + r_2^2 = \tfrac{25}{36} \Leftrightarrow 2r_2^2 + \tfrac{5}{3} r_2 = \tfrac{1}{3} \Leftrightarrow (r_2 + \tfrac{5}{12})^2 = \tfrac{49}{144} \Leftrightarrow r_2 = \tfrac{1}{6} .$$



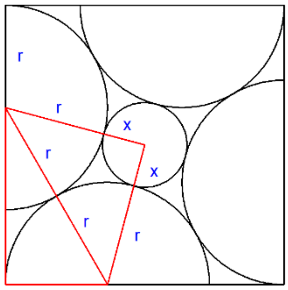
The following relationship exists between the two radii r_1 and r_2 (distance of the circle with radius r_2 from the lower side of the square): $1-2r_2=\frac{1}{2}-2r_1$, thus $r_2=r_1+\frac{1}{4}$

In the right-angled triangle below, the following applies: $r_1^2+x^2=(\frac{1}{2}-r_1)^2$, thus $x^2=(\frac{1}{2}-r_1)^2-r_1^2=\frac{1}{4}-r_1$.

The following applies to the next right-angled triangle lying above: $(1-r_2-r_1)^2+x^2=(r_1+r_2)^2$.

If we insert $r_2=r_1+\frac{1}{4}$, we get: $(1-(r_1+\frac{1}{4})-r_1)^2+x^2=(r_1+r_1+\frac{1}{4})^2$, so we have $(\frac{3}{4}-2r_1)^2+x^2=(2r_1+\frac{1}{4})^2$ and further $\frac{9}{16}-3r_1+4r_1^2+x^2=4r_1^2+r_1+\frac{1}{16} \Leftrightarrow x^2=4r_1-\frac{1}{2}$

From the two equations we get: $x^2=4r_1-\frac{1}{2}=\frac{1}{4}-r_1$, thus $5r_1=\frac{3}{4}$, i.e. $r_1=\frac{3}{20}=0.15$ and further $r_2=0.4$ and $x^2=4\cdot 0.15-\frac{1}{2}=0.1$, therefore $x=\sqrt{0.1}\approx 0.316$.

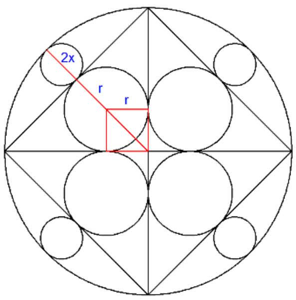


For the left right-angled triangle we have $r^2+(1-r)^2=(2r)^2$, i.e. $r^2+r=\frac{1}{2}$ and further $(r+\frac{1}{2})^2=\frac{1}{2}+\frac{1}{4}$, i.e. $r=\frac{1}{2}\cdot(\sqrt{3}-1)\approx 0.366$.

Fort he right-angled triangle above we have:

$(x+r)^2+(x+r)^2=(2r)^2$, thus $(x+r)^2=2r^2$ and therefore $x=(\sqrt{2}-1)\cdot r$ and

$x=(\sqrt{2}-1)\cdot\frac{1}{2}\cdot(\sqrt{3}-1)\approx 0.152$

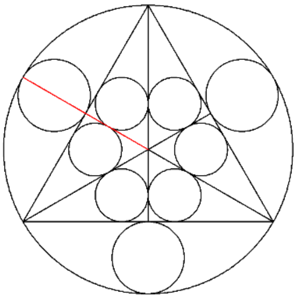


The inscribed square has the side length $\sqrt{2}$. For half the side length of the square and the circles inside it applies $r+r\cdot\sqrt{2}=\frac{1}{2}\cdot\sqrt{2}$, therefore

$$r=\frac{\sqrt{2}}{2\cdot(\sqrt{2}+1)}=\frac{\sqrt{2}\cdot(\sqrt{2}-1)}{2}=1-\frac{1}{2}\cdot\sqrt{2}\approx 0.293$$

Because of $1 = 2x + \frac{1}{2} \cdot \sqrt{2}$ it applies for the radius *x* of the circles outside the triangle :

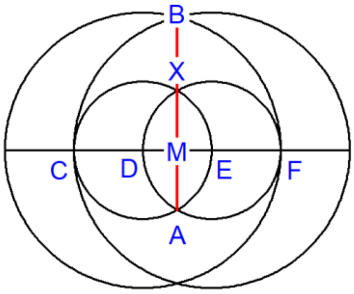
$x = \frac{1}{2} - \frac{1}{4} \cdot \sqrt{2} \approx 0.146$, i.e., this radius is half that of the inner circles.



The diameter 2r of the circles lying outside the equilateral triangle is half the radius of the initial circle, i.e. the radius r of these circles is r = ¼.

From the radius 1 of the circumcircle of the equilateral triangle with side length s results in $\frac{2}{3} \cdot \frac{s}{2} \cdot \sqrt{3} = 1$, thus $s = \sqrt{3}$. The following applies to the radius x of an incircle of a right triangle $x = \frac{1}{2} \cdot (a + b - c)$, see A 15.18, here is $a = \frac{1}{2}$, $b = \frac{1}{2} \cdot \sqrt{3}$ and *c* = 1. Therefore we have: $x = \frac{1}{2} \cdot (\frac{1}{2} + \frac{1}{2} \cdot \sqrt{3} - 1) = \frac{1}{4} \cdot (\sqrt{3} - 1) \approx 0.183$.

✱ A 15.20:



The centres of the small circles are D and E. The radius r of these circles is the same as the lengths of CD, DE, EF, DX, XE, AD, AE. The triangle DEX is therefore an equilateral triangle and the following applies:

$|XM| = |AM| = \frac{1}{2} \cdot r \cdot \sqrt{3}$, thus $|AX| = r \cdot \sqrt{3}$.

The centres of the large circles are also D and E. The radius R of these circles is as large as the lengths of CE, DF, EB, DB, namely $R = 2 \cdot r$. In the right-angled triangle MEB applies: , i.e. $(\frac{1}{2} r)^2 + |MB|^2 = (2r)^2$

From this we get $|MB|^2 = \frac{15}{4} r^2$, therefore $|MB| = \frac{\sqrt{15}}{2} r$.

Therefore the following proportions apply:

$$\frac{|AX|}{|XB|} = \frac{2 \cdot |MX|}{|MB| - |MX|} = \frac{r \cdot \sqrt{3}}{\frac{\sqrt{15}}{2} \cdot r - \frac{1}{2} \cdot r \cdot \sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{15} - \sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{5} \cdot \sqrt{3} - \sqrt{3}} = \frac{2}{\sqrt{5} - 1} = \Phi$$ and also

$$\frac{|AX|}{|AB|} = \frac{2 \cdot |MX|}{|MB| + |AM|} = \frac{r \cdot \sqrt{3}}{\frac{\sqrt{15}}{2} \cdot r + \frac{1}{2} \cdot r \cdot \sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{15} + \sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{5} \cdot \sqrt{3} + \sqrt{3}} = \frac{2}{\sqrt{5} + 1} = \frac{1}{\Phi}.$$

Chapter 16

✱ A 16.1:

n	Σk	n^2	Σk^2	$(n+1)\cdot\Sigma k$	$\Sigma(\Sigma k)$	$(n+1)\cdot\Sigma k-\Sigma(\Sigma k)$
1	1	1	1	2	1	1
2	3	4	5	9	4	5
3	6	9	14	24	10	14
4	10	16	30	50	20	30
5	15	25	55	90	35	55
6	21	36	91	147	56	91
7	28	49	140	224	84	140
8	36	64	204	324	120	204

n	n^3	Σk^3	$(n+1)\cdot\Sigma k^3$	$\Sigma(\Sigma k^3$	$(n+1)\cdot\Sigma k^3-\Sigma(\Sigma k^3)$	k^4	Σk^4
1	1	1	2	1	1	1	1
2	8	9	27	10	17	16	17
3	27	36	144	46	98	81	98
4	64	100	500	146	354	256	354
5	125	225	1350	371	979	625	979
6	216	441	3087	812	2275	1296	2275
7	343	784	6272	1596	4676	2401	4676
8	512	1296	11664	2892	8772	4096	8772

✱ A 16.2:

$c(n+1) - c(n) = \alpha \cdot (n + 1)^2 + \beta \cdot (n + 1) + \gamma - \alpha \cdot n^2 - \beta \cdot n - \gamma = 2\alpha \cdot n + (\alpha + \beta)$

If you compare the coefficients with $b(n) = u \cdot n + v$ you get

$\alpha = \frac{1}{2} \cdot u$ and from that $v = \frac{1}{2} \cdot u + \beta \Leftrightarrow \beta = v - \frac{1}{2} \cdot u$

✱ A 16.3:

- **Sum of the first n fourth powers**

Approach with a 5th-degree function: $s(n) = a \cdot n^5 + b \cdot n^4 + c \cdot n^3 + d \cdot n^2 + e \cdot n + f$

The table shows:

$s(0) = 0$; $s(1) = 1$; $s(2) = 17$; $s(3) = 98$; $s(4) = 354$; $s(5) = 979$. Because of $s(0) = 0$ it follows: $f = 0$.

A linear system of equations with 5 equations and 5 variables must be solved.

$$\left| \begin{array}{c} a+b+c+d+e=1 \\ 32a+16b+8c+4d+2e=17 \\ 243a+81b+27c+9d+3e=98 \\ 1024a+256b+64c+16d+4e=354 \\ 3125a+625b+125c+25d+5e=979 \end{array} \right| .$$

The solution is $\left(\frac{1}{5}; \frac{1}{2}; \frac{1}{3}; 0; -\frac{1}{30}\right)$.

Therefore applies:

$$1^4+2^4+\ldots+n^4=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$$

- **Sum of the first n fifth powers**

Approach with a 6th-degree function:

$s(n) = a \cdot n^6 + b \cdot n^5 + c \cdot n^4 + d \cdot n^3 + e \cdot n^2 + f \cdot n + g$

$s(0) = 0$; $s(1) = 1$; $s(2) = 33$; $s(3) = 276$; $s(4) = 1300$; $s(5) = 4425$; $s(6) = 12201$.

Because of $s(0) = 0$ it follows $g = 0$.

A linear system of equations with 6 equations and 6 variables must be solved.

$$\left| \begin{array}{c} a+b+c+d+e+f=1 \\ 64a+32b+16c+8d+4e+2f=33 \\ 729a+243b+81c+27d+9e+3f=276 \\ 4096a+1024b+256c+64d+16e+4f=1300 \\ 15625a+3125b+625c+125d+25e+5f=4425 \\ 46656a+7776b+1296c+216d+36e+6f=12201 \end{array} \right| .$$

The solution is $\left(\frac{1}{6}; \frac{1}{2}; \frac{5}{12}; 0; -\frac{1}{12}; 0\right)$.

Therefore applies:

$$1^5+2^5+3^5+\ldots+n^5=\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2$$

✱ A 16.4:

- **Sum of the first n cube numbers**

$[\, a \cdot n^4 + b \cdot n^3 + c \cdot n^2 + d \cdot n + e \,] + (n+1)^3 = a \cdot (n+1)^4 + b \cdot (n+1)^3 + c \cdot (n+1)^2 + d \cdot (n+1) + e \;\Leftrightarrow$

$a \cdot n^4 + b \cdot n^3 + c \cdot n^2 + d \cdot n + e + (n^3 + 3 \cdot n^2 + 3 \cdot n + 1) =$

$\qquad a \cdot (n^4 + 4n^3 + 6n^2 + 4n + 1) + b \cdot (n^3 + 3n^2 + 3n + 1) + c \cdot (n^2 + 2n + 1) + d \cdot (n + 1) + e \;\Leftrightarrow$

$n^4 \cdot (a - a) + n^3 \cdot (b + 1 - 4a - b) + n^2 \cdot (c + 3 - 6a - 3b - c) +$
$\qquad n \cdot (d + 3 - 4a - 3b - 2c - d) + (e + 1 - a - b - c - d - e) = 0 \;\Leftrightarrow$

$n^3 \cdot (1 - 4a) + n^2 \cdot (3 - 6a - 3b) + n \cdot (3 - 4a - 3b - 2c) + (1 - a - b - c - d) = 0$

and from that $a = \frac{1}{4}$, $b = \frac{1}{2}$, $c = \frac{1}{4}$, $d = 0$ (and $e = 0$ because of $s(0) = 0$).

- **Sum of the first n fourth powers**

$[\, a \cdot n^5 + b \cdot n^4 + c \cdot n^3 + d \cdot n^2 + e \cdot n + f \,] + (n+1)^4$
$= a \cdot (n+1)^5 + b \cdot (n+1)^4 + c \cdot (n+1)^3 + d \cdot (n+1)^2 + e \cdot (n+1) + f \;\Leftrightarrow$

$a \cdot n^5 + b \cdot n^4 + c \cdot n^3 + d \cdot n^2 + e \cdot n + f + (n^4 + 4 \cdot n^3 + 6 \cdot n^2 + 4 \cdot n + 1)$
$= a \cdot (n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) + b \cdot (n^4 + 4n^3 + 6n^2 + 4n + 1) + c \cdot (n^3 + 3n^2 + 3n + 1)$
$+ d \cdot (n^2 + 2n + 1) + e \cdot (n + 1) + f \;\Leftrightarrow$

$n^5 \cdot (a - a) + n^4 \cdot (b + 1 - 5a - b) + n^3 \cdot (c + 4 - 10a - 4b - c) + n^2 \cdot (d + 6 - 10a - 6b - 3c - d)$
$+ n \cdot (e + 4 - 5a - 4b - 3c - 2d - e) + (f + 1 - a - b - c - d - e - f) = 0 \;\Leftrightarrow$

$n^4 \cdot (1 - 5a) + n^3 \cdot (4 - 10a - 4b) + n^2 \cdot (6 - 10a - 6b - 3c) + n \cdot (4 - 5a - 4b - 3c - 2d) +$
$(1 - a - b - c - d - e) = 0$

and from that $a = \frac{1}{5}$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $d = 0$, $e = -\frac{1}{30}$ (and $f = 0$ because of $s(0) = 0$).

✱ A 16.5:

$$a \cdot n^3 + b \cdot n^2 + c \cdot n + d + \binom{2}{0} \cdot n^2 + \binom{2}{1} \cdot n + \binom{2}{2} \cdot 1$$

$$= a \cdot \left(\binom{3}{0} \cdot n^3 + \binom{3}{1} \cdot n^2 + \binom{3}{2} \cdot n + \binom{3}{3} \right) + b \cdot \left(\binom{2}{0} \cdot n^2 + \binom{2}{1} \cdot n + \binom{2}{2} \right) + c \cdot \left(\binom{1}{0} \cdot n + \binom{1}{1} \right) + d$$

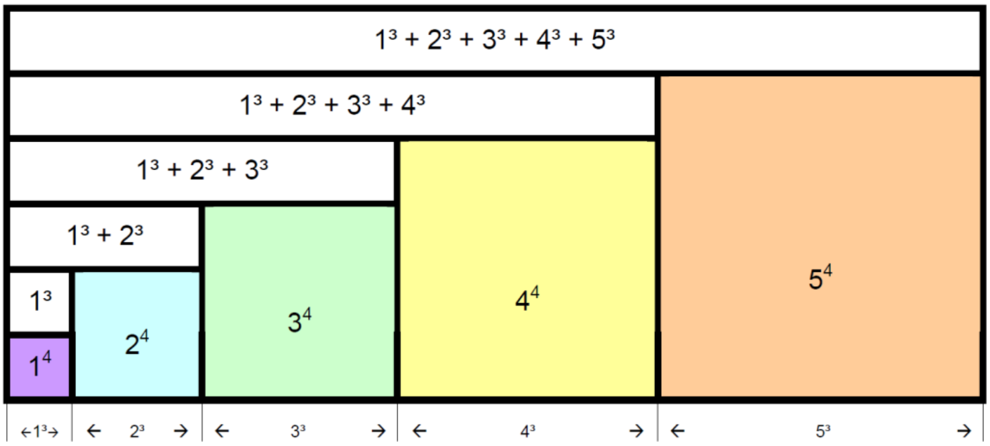
therefore

$$\left(\binom{2}{0} - \binom{3}{1} \cdot a \right) \cdot n^2 + \left(\binom{2}{1} - \binom{3}{2} \cdot a - \binom{2}{1} \cdot b \right) \cdot n + \left(\binom{2}{2} - \binom{3}{3} \cdot a - \binom{2}{2} \cdot b - \binom{1}{1} \cdot c \right) = 0$$

From this we get:

$$\binom{2}{0} - \binom{3}{1} \cdot a = 0 \;\text{ and }\; \binom{2}{1} - \binom{3}{2} \cdot a - \binom{2}{1} \cdot b = 0 \;\text{ and }\; \binom{2}{2} - \binom{3}{3} \cdot a - \binom{2}{2} \cdot b - \binom{1}{1} \cdot c = 0$$

✱ A 16.6:



$$\left[1^4 + 2^4 + 3^4 + \ldots + n^4\right] + \left[1^3 + (1^3 + 2^3) + (1^3 + 2^3 + 3^3) + \ldots + (1^3 + 2^3 + 3^3 + \ldots + n^3)\right]$$
$$= (n+1) \cdot (1^3 + 2^3 + 3^3 + \ldots + n^3)$$

$$\left[1^4+2^4+3^4+\ldots+n^4\right]+\left[\left(\frac{1}{4}\cdot 1^4+\frac{1}{2}\cdot 1^3+\frac{1}{4}\cdot 1^2\right)+\left(\frac{1}{4}\cdot 2^4+\frac{1}{2}\cdot 2^3+\frac{1}{4}\cdot 2^2\right)+\ldots+\left(\frac{1}{4}\cdot n^4+\frac{1}{2}\cdot n^3+\frac{1}{4}\cdot n^2\right)\right]$$

$$=(n+1)\cdot(1^3+2^3+3^3+\ldots+n^3)$$

$$\left[1^4+2^4+3^4+\ldots+n^4\right]+\left[\frac{1}{4}\cdot\left(1^4+2^4+\ldots+n^4\right)+\frac{1}{2}\cdot\left(1^3+2^3+\ldots+n^3\right)+\frac{1}{4}\cdot\left(1^2+2^2+\ldots+n^2\right)\right]$$

$$=(n+1)\cdot(1^3+2^3+3^3+\ldots+n^3)$$

$$\frac{5}{4}\cdot\left(1^4+2^4+\ldots+n^4\right)=\left(n+\frac{1}{2}\right)\cdot(1^3+2^3+3^3+\ldots+n^3)-\frac{1}{4}\cdot\left(1^2+2^2+\ldots+n^2\right)$$

$$1^4+2^4+\ldots+n^4=\frac{4}{5}\cdot\left(n+\frac{1}{2}\right)\cdot(1^3+2^3+3^3+\ldots+n^3)-\frac{4}{5}\cdot\frac{1}{4}\cdot\left(1^2+2^2+\ldots+n^2\right)$$

and further

$$1^4+2^4+\ldots+n^4=\left(\frac{4}{5}n+\frac{2}{5}\right)\cdot\left(\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2\right)-\frac{1}{5}\cdot\left(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\right)$$

$$1^4+2^4+\ldots+n^4=\frac{1}{5}n^5+\frac{2}{5}n^4+\frac{1}{5}n^3+\frac{1}{10}n^4+\frac{1}{5}n^3+\frac{1}{10}n^2-\frac{1}{15}n^3-\frac{1}{10}n^2-\frac{1}{30}n$$

$$1^4+2^4+\ldots+n^4=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$$

✱ A 16.7:

In the first case, the equation can be transformed as follows: $\frac{1^2+2^2+3^2+\ldots+n^2}{1+2+3+\ldots+n}=\frac{2}{3}n+\frac{1}{3}=\frac{2n+1}{3}$.

This is possible because there is a product on the right-hand side of the equation.

Such a transformation is not possible when deriving the sum of the cube numbers because there are two sum terms on the right-hand side of the equation and no simple transformation is possible.

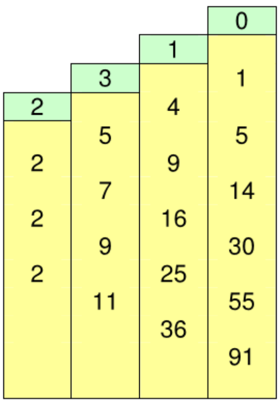
✱ A 16.8:

$$1^2+2^2+3^2+\ldots+n^2$$

$$=0\cdot\binom{n}{0}+1\cdot\binom{n}{1}+3\cdot\binom{n}{2}+2\cdot\binom{n}{3}$$

$$=0+1\cdot\frac{n}{1}+3\cdot\frac{n\cdot(n-1)}{2\cdot 1}+2\cdot\frac{n\cdot(n-1)\cdot(n-2)}{3\cdot 2\cdot 1}=\ldots$$

$$=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$$



✱ A 16.9:

24	60	50	15	1	0
					1
				16	17
			65	81	98
		110	175	256	354
	84	194	369	625	979
24	108	302	671	1296	2275
24	132	434	1105	2401	4676

a	b	c	d	e	f
					f+e
				e+d	f+2e+d
			d+c	e+2d+c	f+3e+3d+c
		c+b	d+2c+b	e+3d+3c+b	f+4e+6d+4c+b
	b+a	c+2b+a	d+3c+3b+a	e+4d+6c+4b+a	f+5e+10d+10c+5b+a
a	b+2a	c+3b+3a	d+4c+6b+4a	e+5d+10c+10b+5a	f+6e+15d+20c+15b+5a
a	b+3a	c+4b+6a	d+5c+10b+10a	e+6d+15c+20b+15a	f+7e+21d+35c+35b+20a

$$1^4+2^4+3^4+\ldots+n^4=0\cdot\binom{n}{0}+1\cdot\binom{n}{1}+15\cdot\binom{n}{2}+50\cdot\binom{n}{3}+60\cdot\binom{n}{4}+24\cdot\binom{n}{5}$$

$$=0+1\cdot\frac{n}{1}+15\cdot\frac{n\cdot(n-1)}{2\cdot 1}+50\cdot\frac{n\cdot(n-1)\cdot(n-2)}{3\cdot 2\cdot 1}+60\cdot\frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}{4\cdot 3\cdot 2\cdot 1}+24\cdot\frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

✱ A 16.10:

- **Sum of the first n cube numbers**

From $n\cdot(n+1)\cdot(n+2)=n^3+3n^2+2n$ we get

$$(1^3+2^3+3^3+\ldots+n^3)+3\cdot(1^2+2^2+3^2+\ldots+n^2)+2\cdot(1+2+3+\ldots+n)=\frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}{4},$$

therefore

$$1^3+2^3+3^3+\ldots+n^3=\frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}{4}-3\cdot\frac{n\cdot(n+1)\cdot(2n+1)}{6}-2\cdot\frac{n\cdot(n+1)}{2}$$

$$=\frac{n\cdot(n+1)}{2}\cdot\left[\frac{(n+2)\cdot(n+3)}{2}-(2n+1)-2\right]=\frac{n\cdot(n+1)}{2}\cdot\frac{n\cdot(n+1)}{2}=\left[\frac{n\cdot(n+1)}{2}\right]^2$$

- **Sum of the first n fourth powers**

From $n\cdot(n+1)\cdot(n+2)\cdot(n+3)=n^4+6n^3+11n^2+6n$ we get

$$1\cdot 2\cdot 3\cdot 4+2\cdot 3\cdot 4\cdot 5+3\cdot 4\cdot 5\cdot 6+\ldots+n\cdot(n+1)\cdot(n+2)\cdot(n+3)=\frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)}{5}$$

$$(1^4+2^4+3^4+\ldots+n^4)+6\cdot(1^3+2^3+3^3+\ldots+n^3)+11\cdot(1^2+2^2+3^2+\ldots+n^2)+6\cdot(1+2+3+\ldots+n)$$
$$=\frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)}{5}$$

therefore

$$(1^4+2^4+3^4+...+n^4)$$
$$=\frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)}{5}-6\cdot(1^3+2^3+3^3+...+n^3)-11\cdot(1^2+2^2+3^2+...+n^2)-6\cdot(1+2+3+...+n)$$
$$=\frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)}{5}-6\cdot\frac{n^2\cdot(n+1)^2}{4}-11\cdot\frac{n\cdot(n+1)\cdot(2n+1)}{6}-6\cdot\frac{n\cdot(n+1)}{2}$$
$$=\frac{n\cdot(n+1)}{30}\cdot\left[6\cdot(n+2)\cdot(n+3)\cdot(n+4)-45\cdot n\cdot(n+1)-55\cdot(2n+1)-90\right]$$
$$=\frac{n\cdot(n+1)}{30}\cdot\left[6n^3+54n^2+156n+144-45n^2-45n-110n-55-90\right]$$
$$=\frac{n\cdot(n+1)}{30}\cdot\left[6n^3+9n^2+n-1\right]=\frac{n\cdot(n+1)}{30}\cdot(2n+1)\cdot(3n^2+3n-1)$$

✱ A 16.11:

- **Sum of the first n cube numbers**

From $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$ we get from an analogously constructed table

$(n + 1)^4 - 1^4 = 4 \cdot (1^3 + 2^3 + 3^3 + \ldots + n^3) + 6 \cdot (1^2 + 2^2 + 3^2 + \ldots + n^2) + 4 \cdot (1 + 2 + 3 + \ldots + n) + n \cdot 1$

therefore

$4 \cdot (1^3 + 2^3 + 3^3 + \ldots + n^3) = (n + 1)^4 - 1^4 - 6 \cdot (1^2 + 2^2 + 3^2 + \ldots + n^2) - 4 \cdot (1 + 2 + 3 + \ldots + n) - n \cdot 1$

$= n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 - 2n^3 - 3n^2 - n - 2n^2 - 2n - n$

$= n^4 + 2n^3 + n^2 = n^2 \cdot (n^2 + 2n + 1) = n^2 \cdot (n + 1)^2$

thus

$(1^3 + 2^3 + 3^3 + \ldots + n^3) = \frac{1}{4} \cdot n^2 \cdot (n + 1)^2$

- **Sum of the first n fourth powers**

From $(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1$ we get from an analogously constructed table

$(n + 1)^5 - 1^5$

$= 5 \cdot (1^4 + 2^4 + ... + n^4) + 10 \cdot (1^3 + 2^3 + \ldots + n^3) + 10 \cdot (1^2 + 2^2 + \ldots + n^2) + 5 \cdot (1 + 2 + \ldots + n) + n \cdot 1$

therefore

$5 \cdot (1^4 + 2^4 + ... + n^4)$

$= (n + 1)^5 - 1^5 - 10 \cdot (1^3 + 2^3 + \ldots + n^3) - 10 \cdot (1^2 + 2^2 + \ldots + n^2) - 5 \cdot (1 + 2 + \ldots + n) - n \cdot 1$

$= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - 1 - 2{,}5n^4 - 5n^3 - 2{,}5n^2 - 10/3 \cdot n^3 - 5n^2 - 5/3 \cdot n - 2{,}5n^2 - 2{,}5n - n$

$= n^5 + 5/2 \cdot n^4 + 5/3 \cdot n^3 - 1/6 \cdot n$

thus

$$1^4+2^4+3^4+...+n^4=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$$

✱ A 16.12:

$$S_4(n)=1^4+2^4+3^4+...+n^4=\frac{1}{5}\cdot\left[\binom{5}{0}\cdot B_0\cdot n^5+\binom{5}{1}\cdot B_1\cdot n^4+\binom{5}{2}\cdot B_2\cdot n^3+\binom{5}{3}\cdot B_3\cdot n^2+\binom{5}{4}\cdot B_4\cdot n^1\right]$$
$$=\frac{1}{4}\cdot\left[B_0\cdot n^5+5\cdot B_1\cdot n^4+10\cdot B_2\cdot n^3+10\cdot B_3\cdot n^2+5\cdot B_4\cdot n^1\right]=\frac{1}{5}\cdot\left[n^5+\frac{5}{2}n^4+\frac{5}{3}n^3-\frac{1}{6}n^1\right]$$

with $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$ and $B_4 = -\frac{1}{30}$.

$$S_5(n) = 1^5 + 2^5 + 3^5 + \ldots + n^5$$
$$= \frac{1}{6} \cdot \left[\binom{6}{0} \cdot B_0 \cdot n^6 + \binom{6}{1} \cdot B_1 \cdot n^5 + \binom{6}{2} \cdot B_2 \cdot n^4 + \binom{6}{3} \cdot B_3 \cdot n^3 + \binom{6}{4} \cdot B_4 \cdot n^2 + \binom{6}{5} \cdot B_5 \cdot n^1 \right]$$
$$= \frac{1}{6} \cdot \left[B_0 \cdot n^6 + 6 \cdot B_1 \cdot n^5 + 15 \cdot B_2 \cdot n^4 + 20 \cdot B_3 \cdot n^3 + 15 \cdot B_4 \cdot n^2 + 6 \cdot B_5 \cdot n^1 \right]$$
$$= \frac{1}{6} \cdot \left[n^6 + 3n^5 + \frac{5}{2} n^4 - \frac{1}{2} n^2 \right]$$

with $B_0 = 1$, $B_1 = ½$, $B_2 = {}^1/_6$, $B_3 = 0$, $B_4 = -{}^1/_{30}$, $B_5 = 0$

✱ A 16.13:

- **Sum of the first n square numbers**

Approach with a 3rd degree function and the points P_0 (0|0), P_1 (1|1), P_2 (2|5), P_3 (3|14):

$$L_3(n) = \frac{(n-x_1)(n-x_2)(n-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(n-x_0)(n-x_2)(n-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$
$$+ \frac{(n-x_0)(n-x_1)(n-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(n-x_0)(n-x_1)(n-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$
$$= \frac{(n-1)(n-2)(n-3)}{(0-1)(0-2)(0-3)} \cdot 0 + \frac{(n-0)(n-2)(n-3)}{(1-0)(1-2)(1-3)} \cdot 1$$
$$+ \frac{(n-0)(n-1)(n-3)}{(2-0)(2-1)(2-3)} \cdot 5 + \frac{(n-0)(n-1)(n-2)}{(3-0)(3-1)(3-2)} \cdot 14$$
$$= 0 + \frac{1}{2} \cdot (n^3 - 5n^2 + 6n) - \frac{5}{2} \cdot (n^3 - 4n^2 + 3n) + \frac{14}{6} \cdot (n^3 - 3n^2 + 2n)$$
$$= \frac{1}{3} \cdot n^3 + \frac{1}{2} \cdot n^2 + \frac{1}{6} \cdot n$$

- **Sum of the first n cube numbers**

Approach with a 4th degree function and the points P_0 (0|0), P_1 (1|1), P_2 (2|9), P_3 (3|36), P_4 (4|100):

$$L_4(n) = \frac{(n-1)(n-2)(n-3)(n-4)}{(0-1)(0-2)(0-3)(0-4)} \cdot 0 + \frac{(n-0)(n-2)(n-3)(n-4)}{(1-0)(1-2)(1-3)(1-4)} \cdot 1$$
$$+ \frac{(n-0)(n-1)(n-3)(n-4)}{(2-0)(2-1)(2-3)(2-4)} \cdot 9 + \frac{(n-0)(n-1)(n-2)(n-4)}{(3-0)(3-1)(3-2)(3-4)} \cdot 36 + \frac{(n-0)(n-1)(n-2)(n-3)}{(4-0)(4-1)(4-2)(4-3)} \cdot 100$$
$$= 0 - \frac{1}{6} \cdot (n^4 - 9n^3 + 26n^2 - 24n) + \frac{9}{4} \cdot (n^4 - 8n^3 + 19n^2 - 12n)$$
$$- \frac{36}{6} \cdot (n^4 - 7n^3 + 14n^2 - 8n) + \frac{100}{24} \cdot (n^4 - 6n^3 + 11n^2 - 6n)$$
$$= \frac{1}{4} \cdot n^4 + \frac{1}{2} \cdot n^3 + \frac{1}{4} \cdot n^2$$

Chapter 17

✱ A 17.1:

Figure 1 and 3: Application of the Pythagorean theorem to the right-angled triangle whose hypotenuse is a side of the large triangle and whose sides in the large triangle are altitude and hypotenuse segment:

$a^2 = h^2 + p^2$ and $b^2 = h^2 + q^2$

Figure 2 and 4: Since according to Euclid's theorem the square on one leg is equal in area to the rectangle of the corresponding segment of the hypotenuse and the hypotenuse ($a^2 = c \cdot p$ and $b^2 = c \cdot q$), it follows that the square above the altitude is equal in area to the two remaining rectangles coloured purple:

$h^2 = p \cdot (c - p)$ and $h^2 = q \cdot (c - q)$

(since $c = p + q$ this is nothing else but the statement of Euclid's right triangle altitude theorem)

Figure 5: The total area of the two squares on the sides can thus be represented as the sum of the areas of the two squares above the hypotenuse segments and two mutually congruent residual rectangles that are equal in area to the square on the altitude: $a^2 + b^2 = p^2 + q^2 + 2h^2$

Of course, this can also be derived by (algebraic) transformation:

$a^2 + b^2 = c^2 = (p + q)^2 = p^2 + 2pq + q^2 = p^2 + 2h^2 + q^2$

✱ A 17.2:

(1) The points and sides of the figure to be examined are labelled as can be seen on the right.

The figure is defined by the right-angled triangle ABC, i.e. by the right angle at C and the side lengths a and b. The figure is then divided into two parts.

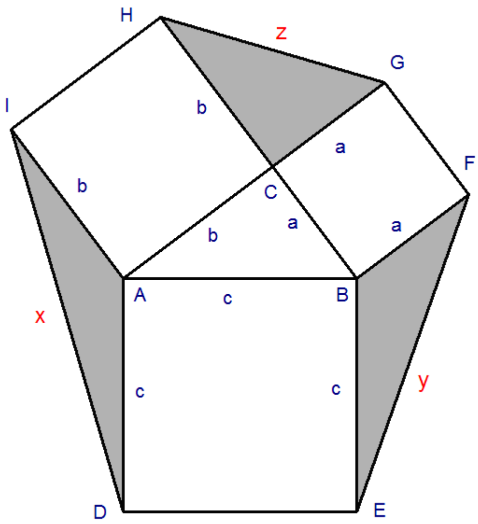
From this we can calculate: $c = \sqrt{a^2 + b^2}$

$\alpha = \tan^{-1}(a/b)$ and $\beta = 90° - \alpha$

It applies: $\sin(\alpha) = \frac{a}{c}$ und $\cos(\alpha) = \frac{b}{c}$

The **triangle ABC** has the area $A_1 = ½ \cdot a \cdot b$

Triangle CGH:
The triangle is congruent to the triangle ABC.



Triangle AID: The sides b and c of the triangle are given as well as the angle
$\delta = \angle(DAI) = 360° - (90° + 90° + \alpha) = 180° - \alpha$.

Because of $\sin(180° - \alpha) = \sin(\alpha)$ and $a = c \cdot \sin(\alpha)$ we get the area

$A_2 = ½ \cdot b \cdot c \cdot \sin(\delta) = ½ \cdot b \cdot c \cdot \sin(\alpha) = ½ \cdot b \cdot a = A_1$

Triangle BEF:

The sides a and c are given also the angle
$\varepsilon = \angle(FBE) = 360° - (90° + 90° + \beta) = 180° - \beta = 180° - (90° - \alpha) = 90° + \alpha$.

Because of $\sin(90° + \alpha) = \cos(\alpha)$ und $b = c \cdot \cos(\alpha)$ we get the area

$A_3 = ½ \cdot a \cdot c \cdot \sin(\varepsilon) = ½ \cdot a \cdot c \cdot \cos(\alpha) = ½ \cdot a \cdot b = A_1$

(2) The length of side x can be calculated with help of the law of cosinus:

$x^2 = b^2 + c^2 - 2bc \cdot \cos(\delta) = b^2 + c^2 + 2bc \cdot \cos(\alpha) = b^2 + (a^2 + b^2) + 2b^2$, thus $\boxed{x^2 = a^2 + 4b^2}$

and because $\cos(180° - \alpha) = -\cos(\alpha)$ and $b = c \cdot \cos(\alpha)$

The length of side y can also be calculated with help oft he law of cosinus:

$y^2 = a^2 + c^2 - 2ac \cdot \cos(\varepsilon) = a^2 + c^2 - 2ac \cdot \cos(90° + \alpha)$

$= a^2 + c^2 + 2ac \cdot \sin(\alpha) = a^2 + (a^2 + b^2) + 2a^2$, also $\boxed{y^2 = 4a^2 + b^2}$

and because $\cos(90° + \alpha) = -\sin(\alpha)$ and $a = c \cdot \sin(\alpha)$

Because of $z = c$ and $c^2 = a^2 + b^2$ we get:

$x^2 + y^2 + z^2 = a^2 + 4b^2 + 4a^2 + b^2 + a^2 + b^2 = 6 \cdot (a^2 + b^2) = 3 \cdot (a^2 + b^2) + 3 \cdot (a^2 + b^2) = 3 \cdot (a^2 + b^2 + c^2)$

(3) This is exactly what has been proven in (2).

(4) Compared to equation (1) the calculation of the area of the triangles ABC and CGH changes.

For the **triangle ABC** we have: $A_1 = ½ \cdot a \cdot b \cdot \sin(\gamma) = ½ \cdot b \cdot c \cdot \sin(\alpha) = ½ \cdot c \cdot a \cdot \sin(\beta)$

In the **triangle CGH** we have for the angle at point C: $\varphi = \angle(GCH) = 360° - (90° + 90° + \gamma) = 180° - \gamma$.

Therefore we get for the area: $A_4 = ½ \cdot a \cdot b \cdot \sin(\varphi) = ½ \cdot a \cdot b \cdot \sin(\gamma) = A_1$

In the triangle **AID** we have: $A_2 = ½ \cdot b \cdot c \cdot \sin(\delta) = ½ \cdot b \cdot c \cdot \sin(\alpha) = A_1$

In the triangle **BEF** we have: $A_3 = ½ \cdot a \cdot c \cdot \sin(\varepsilon) = ½ \cdot a \cdot c \cdot \cos(\alpha) = A_1$

Compared to equation (2) the calculation of x^2, y^2 und z^2 changes:

$x^2 = b^2 + c^2 - 2bc \cdot \cos(\delta) = b^2 + c^2 - 2bc \cdot \cos(180° - \alpha) = b^2 + c^2 + 2bc \cdot \cos(\alpha)$

$y^2 = a^2 + c^2 - 2ac \cdot \cos(\varepsilon) = a^2 + c^2 - 2ac \cdot \cos(180° - \beta) = a^2 + c^2 + 2ac \cdot \cos(\beta)$

$z^2 = a^2 + b^2 - 2ab \cdot \cos(\varphi) = a^2 + b^2 - 2ab \cdot \cos(180° - \gamma) = a^2 + b^2 + 2ab \cdot \cos(\gamma)$

Because of $a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha)$, i.e. $2bc \cdot \cos(\alpha) = b^2 + c^2 - a^2$, and analogously we have

$2ca \cdot \cos(\beta) = c^2 + a^2 - b^2$, $2ab \cdot \cos(\gamma) = a^2 + b^2 - c^2$:

$x^2 + y^2 + z^2 = 2a^2 + 2b^2 + 2c^2 + b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2 = 3a^2 + 3b^2 + 3c^2$

✱ A 17.3:

As explained in the solution of A 17.2, it applies for the yellow triangle below: $A_u = ½ \cdot a \cdot b \cdot \sin(\gamma)$

and for the yellow triangle above (with the angle $\varphi = 360° - (90° + 90° + \gamma) = 180° - \gamma$):

$A_o = ½ \cdot a \cdot b \cdot \sin(\varphi) = ½ \cdot a \cdot b \cdot \sin(\gamma) = A_u$

With the designations of A 17.2 we get:

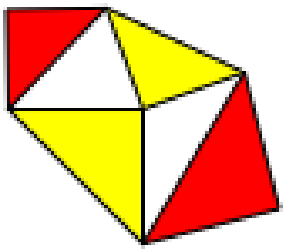
$c^2 + z^2 = c^2 + a^2 + b^2 + 2ab \cdot \cos(\gamma) = c^2 + a^2 + b^2 + a^2 + b^2 - c^2 = 2a^2 + 2b^2 = 2 \cdot (a^2 + b^2)$

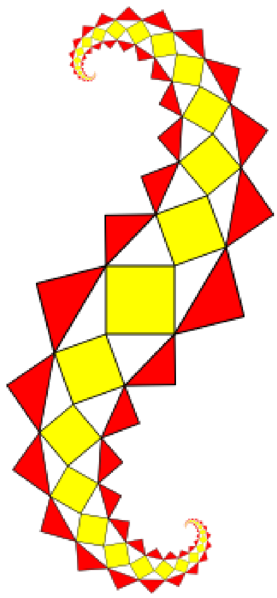
Tip: If instead of the green squares we look at the isosceles right-angled triangles half the size of the green squares and instead of the light blue squares we look at a quarter of each of them, these are isosceles right-angled triangles, then the following figure by Hans Walser results

red = yellow

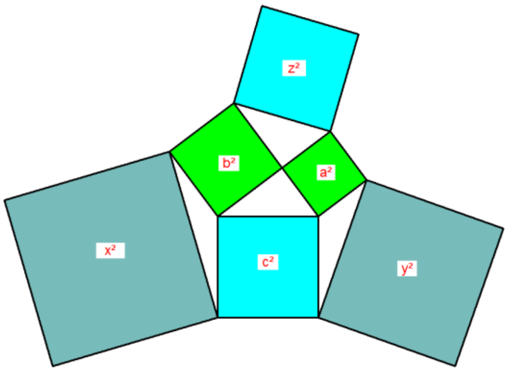
see also

www.walser-h-m.ch/hans/Miniaturen/D/Drachenspirale/Drachenspirale.htm





The figures in Fig. 17.2b and Fig. 17.2c can be completed in different ways, for example like this:



Above we have shown: $c^2 + z^2 = 2 \cdot (a^2 + b^2)$

Analogously we have in the left figure:

$x^2 + a^2 = 2 \cdot (b^2 + c^2)$ und

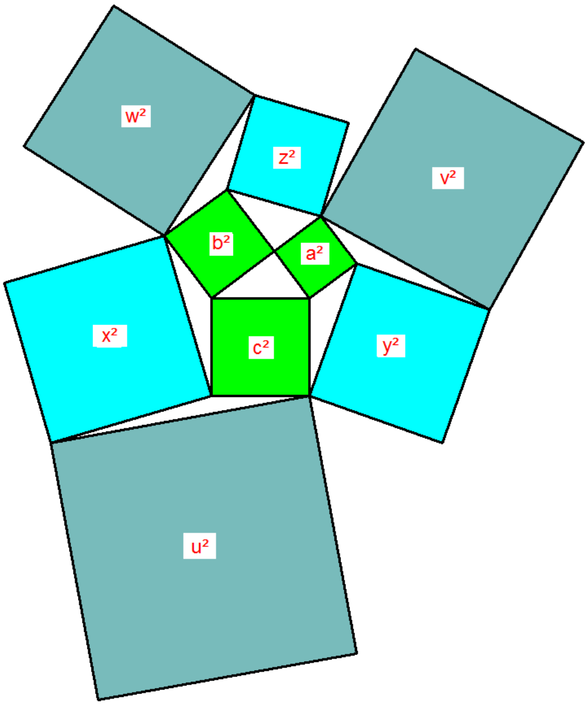
$y^2 + b^2 = 2 \cdot (a^2 + c^2)$

Therefore we get:

$x^2 + y^2 = 2 \cdot (b^2 + c^2 + a^2 + c^2) - a^2 - b^2$

$= b^2 + a^2 + 4c^2 = a^2 + b^2 + 4 \cdot (a^2 + b^2)$

$= 5 \cdot (a^2 + b^2)$



Above we have shown:

$x^2 + y^2 + z^2 = 3 \cdot (a^2 + b^2 + c^2)$

Here we have

$u^2 + b^2 = 2 \cdot (x^2 + c^2)$ and

$w^2 + a^2 = 2 \cdot (b^2 + z^2)$ and

$v^2 + c^2 = 2 \cdot (a^2 + y^2)$,

thus

$u^2 + v^2 + w^2 + a^2 + b^2 + c^2$

$= 2 \cdot (x^2 + c^2) + 2 \cdot (z^2 + b^2) + 2 \cdot (y^2 + a^2)$,

i.e.

$u^2 + v^2 + w^2$

$= 2 \cdot x^2 + c^2 + 2 \cdot z^2 + b^2 + 2 \cdot y^2 + a^2$

$= 2 \cdot (x^2 + y^2 + z^2) + (a^2 + b^2 + c^2)$

$= 7 \cdot (a^2 + b^2 + c^2)$

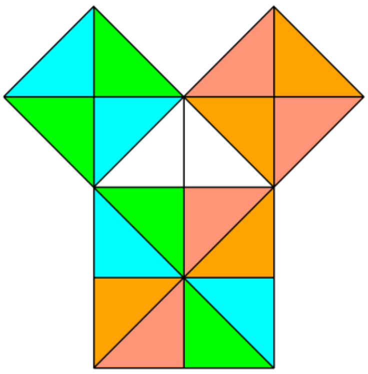
Please also look at:

http://www.walser-h-m.ch/hans/Miniaturen/Q/Quadrate_ansetzen/Quadrate_ansetzen.htm

http://www.walser-h-m.ch/hans/Miniaturen/Q/Quadrate_ansetzen2/Quadrate_ansetzen2.htm

http://www.walser-h-m.ch/hans/Miniaturen/P/Pythagoras-Schmetterling/Pythagoras-Schmetterling.htm

✱ A 17.4:



✱ A 17.5:

Draw the smaller square on the leg with the area a^2 at the bottom left into the larger square of the leg with the area b^2. The remaining area with the area $b^2 – a^2$ is divided into the light blue coloured square with the area $(b – a)^2$ and the two congruent rectangles with the area $(b – a) \cdot a$.

The lower red line separates partial areas:

– a green coloured trapezoid of width b – a and the two sides parallel to each other with the side lengths a and a^2/b

– a small green coloured triangle with the sides b – a and $a – a^2/b$

– a yellow-coloured triangle with the sides a and $a – a^2/b$

– a yellow-coloured trapezoid with the parallel sides a and a^2/b and the height a

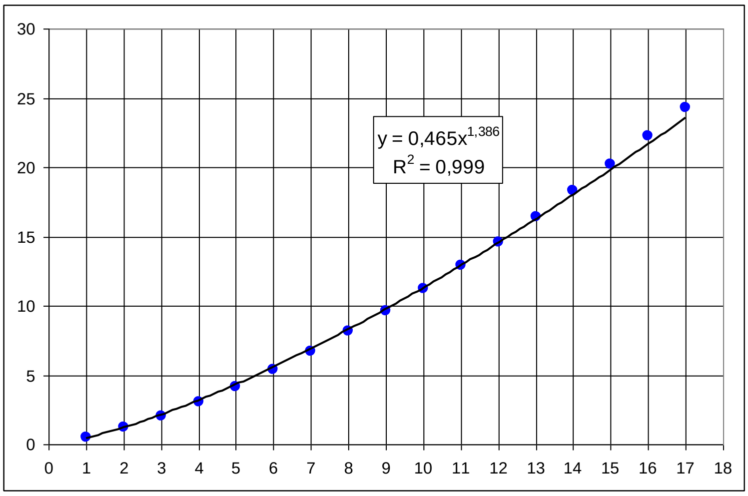
These four coloured puzzle pieces and the light blue coloured square are placed in the square on the hypotenuse. The smaller yellow coloured square and the right green and light blue-coloured rectangular strip of the larger square have been used. The left part of the square (coloured white and yellow in the third illustration) is divided by the upper red line into two congruent right-angled triangles, which fill the rest of the square on the hypotenuse without being coloured.

✱ A 17.6:

The areas of the triangles are calculated according to the formula $A_n = \frac{1}{2} \cdot 1 \cdot \sqrt{n}$. With the 16th triangle there is not yet a full circle, with the 17th the total angle of 360° is exceeded. These angles result for n = 1, 2, ... from $\tan(\alpha) = \frac{1}{\sqrt{n}}$

n	area	area cumulated	angle	angle cumulated
1	0,500	0,500	45,000	45,000
2	0,707	1,207	35,264	80,264
3	0,866	2,073	30,000	110,264
4	1,000	3,073	26,565	136,829
5	1,118	4,191	24,095	160,924
6	1,225	5,416	22,208	183,132
7	1,323	6,739	20,705	203,837
8	1,414	8,153	19,471	223,308
9	1,500	9,653	18,435	241,743
10	1,581	11,234	17,548	259,291
11	1,658	12,892	16,779	276,070
12	1,732	14,625	16,102	292,172
13	1,803	16,427	15,501	307,673
14	1,871	18,298	14,963	322,637
15	1,936	20,235	14,478	337,114
16	2,000	22,235	14,036	351,150
17	2,062	24,296	13,633	364,783
18	2,121	26,417	13,263	378,046

The total area of the Pythagorean spiral can (for $n < 18$) be modelled approximately by the power function f with $f(x) = 0{,}465 \cdot x^{1{,}386}$ (cf. EXCEL sheet with coefficient of determination 99.9 %).



✱ A 17.7:

First possibilty:

Start left with the right-angled triangle with the sides a_0, b_0, c_0, then we have $a_0^2 + b_0^2 = c_0^2$. If one designates the right-angled triangles that become smaller towards the right in the same way, i.e. a_1, b_1, c_1, a_2, b_2, c_2 etc., then we have: $b_{n+1} = a_n$.

Therefore it follows: $a_{n+1} : b_{n+1} = a_n : b_n$, thus:

$a_{n+1} = b_{n+1} \cdot \frac{a_n}{b_n} = \frac{a_n}{b_n} \cdot a_n$ and analogously: $c_{n+1} = \frac{a_n}{b_n} \cdot c_n$, therefore we get

$$a_1 = \frac{a_0^2}{b_0} \;;\; a_2 = a_1^2 \cdot \frac{1}{b_1} = \left(\frac{a_0^2}{b_0}\right)^2 \cdot \frac{1}{a_0} = \frac{a_0^3}{b_0^2} \;;\; a_3 = a_2^2 \cdot \frac{1}{b_2} = \left(\frac{a_0^3}{b_0^2}\right)^2 \cdot \frac{1}{a_1} = \frac{a_0^6}{b_0^4} \cdot \frac{b_0}{a_0^2} = \frac{a_0^4}{b_0^3} \;;$$

$a_4 = a_3^2 \cdot \frac{1}{b_3} = \left(\frac{a_0^4}{b_0^3}\right)^2 \cdot \frac{1}{a_2} = \frac{a_0^8}{b_0^6} \cdot \frac{b_0^2}{a_0^3} = \frac{a_0^5}{b_0^4}$ etc., i.e. starting with a_0 we get the elements of a sequence by multiplication with the constant factor $q = \frac{a_0}{b_0}$. This also results from considerations about similarity.

Further applies: $c_1 = \frac{a_0}{b_0} \cdot c_0$; $c_2 = \frac{a_1}{b_1} \cdot c_1 = \frac{a_0^2}{b_0} \cdot \frac{1}{a_0} \cdot \frac{a_0}{b_0} \cdot c_0 = \frac{a_0^2}{b_0^2} \cdot c_0$;

$c_3 = \frac{a_2}{b_2} \cdot c_2 = \frac{a_0^3}{b_0^2} \cdot \frac{1}{a_1} \cdot \frac{a_0^2}{b_0^2} \cdot c_0 = \frac{a_0^3}{b_0^2} \cdot \frac{b_0}{a_0^2} \cdot \frac{a_0^2}{b_0^2} \cdot c_0 = \frac{a_0^3}{b_0^3} \cdot c_0$ etc., i.e. starting with c_0 we get the elements of a sequence by multiplication with the constant factor $q = \frac{a_0}{b_0}$.

This also applies for the other side: $b_1 = a_0$; $b_2 = a_1 = \frac{a_0^2}{b_0}$; $b_3 = a_2 = \frac{a_0^3}{b_0^2}$ etc., i.e. also the other sides are elements of a sequence with constant factor $q = \frac{a_0}{b_0}$.

Therefore applies for the areas of the triangles:

$$A_T = \frac{1}{2}\cdot a_0\cdot b_0 + \frac{1}{2}\cdot a_1\cdot b_1 + \frac{1}{2}\cdot a_2\cdot b_2 + \frac{1}{2}\cdot a_3\cdot b_3 + ...$$

$$= \frac{1}{2}\cdot\left[a_0\cdot b_0 + \left(\frac{a_0}{b_0}\right)\cdot a_0\cdot a_0 + \left(\frac{a_0}{b_0}\right)^2\cdot a_0\cdot\left(\frac{a_0}{b_0}\right)\cdot a_0 + \left(\frac{a_0}{b_0}\right)^3\cdot a_0\cdot\left(\frac{a_0}{b_0}\right)^2\cdot a_0 + ...\right]$$

$$= \frac{1}{2}\cdot\left[a_0\cdot b_0 + \left(\frac{a_0}{b_0}\right)^1\cdot a_0^{\ 2} + \left(\frac{a_0}{b_0}\right)^3\cdot a_0^{\ 2} + \left(\frac{a_0}{b_0}\right)^5\cdot a_0^{\ 2} + ...\right]$$

$$= \frac{1}{2}\cdot a_0\cdot b_0 + \frac{1}{2}\cdot a_0^{\ 2}\cdot\left[q^1 + q^3 + q^5 + ...\right] = \frac{1}{2}\cdot a_0\cdot b_0 + \frac{1}{2}\cdot a_0^{\ 2}\cdot q\cdot\left[1 + q^2 + q^4 + ...\right]$$

$$= \frac{1}{2}\cdot a_0\cdot b_0 + \frac{1}{2}\cdot a_0^{\ 2}\cdot q\cdot\frac{1}{1-q^2} = \frac{1}{2}\cdot a_0\cdot b_0 + \frac{1}{2}\cdot a_0^{\ 2}\cdot\frac{a_0}{b_0}\cdot\frac{1}{1-\left(\frac{a_0}{b_0}\right)^2}$$

$$= \frac{1}{2}\cdot a_0\cdot b_0 + \frac{1}{2}\cdot a_0^{\ 2}\cdot\frac{a_0}{b_0}\cdot\frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}} = \frac{1}{2}\cdot a_0\cdot b_0 + \frac{1}{2}\cdot\frac{a_0^{\ 3}\cdot b_0}{b_0^{\ 2}-a_0^{\ 2}} = \frac{1}{2}\cdot a_0\cdot b_0\cdot\left[1 + \frac{a_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}}\right]$$

$$= \frac{1}{2}\cdot a_0\cdot b_0\cdot\frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}}$$

And for the areas of the inside lying squares we have:

$$A_{Sq-inside} = b_0^{\ 2} + b_1^{\ 2} + b_2^{\ 2} + b_3^{\ 2} + b_4^{\ 2} + ... = b_0^{\ 2} + a_0^{\ 2} + a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2} + ...$$

$$= b_0^{\ 2} + a_0^{\ 2} + \left(q\cdot a_0\right)^2 + \left(q^2\cdot a_0\right)^2 + \left(q^3\cdot a_0\right)^2 + ... = b_0^{\ 2} + a_0^{\ 2} + q^2\cdot a_0^{\ 2} + q^4\cdot a_0^{\ 2} + q^6\cdot a_0^{\ 2} + ...$$

$$= b_0^{\ 2} + a_0^{\ 2}\cdot\frac{1}{1-q^2} = b_0^{\ 2} + a_0^{\ 2}\cdot\frac{1}{1-\left(\frac{a_0}{b_0}\right)^2} = b_0^{\ 2} + a_0^{\ 2}\cdot\frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}} = b_0^{\ 2}\cdot\left[1 + \frac{a_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}}\right]$$

$$= b_0^{\ 2}\cdot\frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}} = \frac{b_0^{\ 4}}{b_0^{\ 2}-a_0^{\ 2}}$$

And for the areas of the outside lying squares we have:

$$A_{Sq-outside} = c_0^{\ 2} + c_1^{\ 2} + c_2^{\ 2} + c_3^{\ 2} + c_4^{\ 2} + ... = c_0^{\ 2} + \left(q\cdot c_0\right)^2 + \left(q^2\cdot c_0\right)^2 + \left(q^3\cdot c_0\right)^2 + \left(q^4\cdot c_0\right)^2 + ...$$

$$= c_0^{\ 2}\cdot\frac{1}{1-q^2} = c_0^{\ 2}\cdot\frac{1}{1-\left(\frac{a_0}{b_0}\right)^2} = c_0^{\ 2}\cdot\frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}} = \left(a_0^{\ 2}+b_0^{\ 2}\right)\cdot\frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}}$$

Thus we have in total:

$$A = 2\cdot A_D + 2\cdot A_{Qa} + A_{Qi} = a_0\cdot b_0\cdot\frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}} + 2\cdot\frac{b_0^{\ 4}}{b_0^{\ 2}-a_0^{\ 2}} + \left(a_0^{\ 2}+b_0^{\ 2}\right)\cdot\frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}}$$

$$= \frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}}\cdot\left[a_0\cdot b_0 + 2\cdot b_0^{\ 2} + a_0^{\ 2} + b_0^{\ 2}\right] = \frac{b_0^{\ 2}}{b_0^{\ 2}-a_0^{\ 2}}\cdot\left[a_0^{\ 2} + a_0\cdot b_0 + 3\cdot b_0^{\ 2}\right]$$

Second possibity

Use the relationship from A 17.3. Here it was shown that two of the squares coloured violet are twice as large as two of the squares coloured light blue.

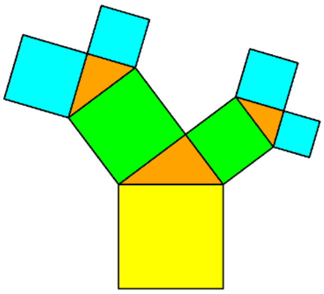
✱ A 17.8:

The entire figure is defined by the lengths of the sides of the initial figure, i.e. by a and b:

In the first step, the whole figure consists of the square on the hypotenuse with area $c^2 = a^2 + b^2$, the right-angled triangle with area $\frac{1}{2} \cdot a \cdot b$ and the two squares on the sides, i.e. $A_1 = 2 \cdot (a^2 + b^2) + \frac{1}{2} \cdot a \cdot b$

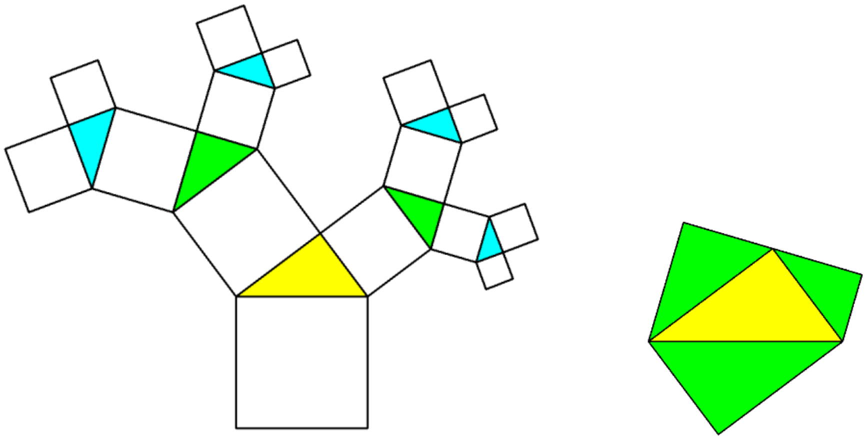
In the next step, a right-angled triangle and two new squares on the sides are added to the left and to the right respectively, the areas of each of which are of the same size as the squares on the hypotenuse according to the Pythagorean theorem.

From the following diagram it is clear that the four squares of the 2nd step coloured light blue are as large as the two squares of the 1st step coloured green, which in turn are together as large as the square on the hypotenuse of the 1st step.

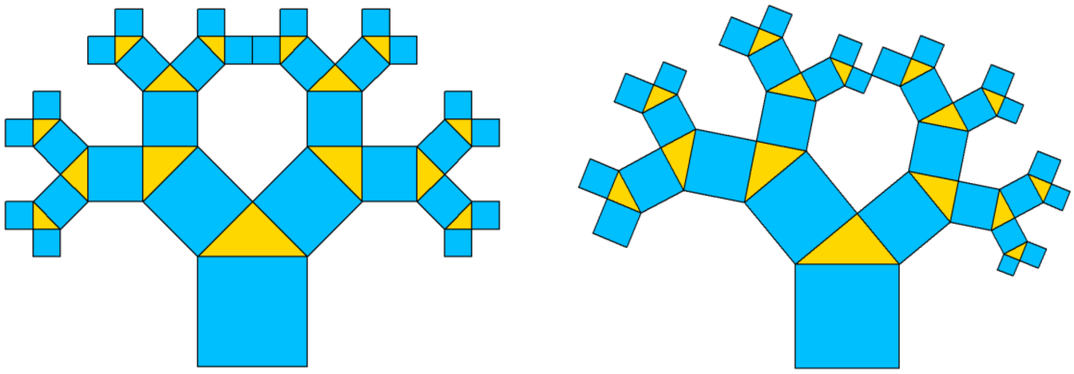


Since in the first step the squares already have an area of $2 \cdot (a^2 + b^2)$ and with each further step an area of $a^2 + b^2$ is added, the total area of the squares after n steps is $(n + 1) \cdot (a^2 + b^2)$.

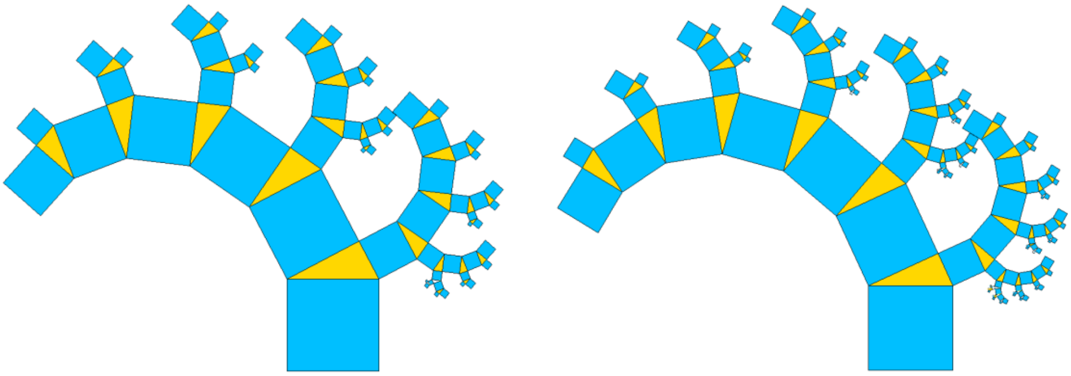
If you omit all the squares in a Pythagorean tree, the result is a tree of right-angled triangles. Here, the two triangles of the 2nd step together are as large as the triangle of the 1st step, cf. the figure on the right and the explanations in section 17.7, so the total area of the triangles after n steps is $n \cdot \frac{1}{2} \cdot a \cdot b$.



The total area of the figure thus grows with n beyond all limits – if one does not consider that the branches already overlap after a few steps.



For a = b, i.e. $\alpha = 45°$, squares will touch each other at the 3rd step of the figure. For $\alpha \approx 39.4°$, i.e. a ≈ 0.821 · b, squares will touch each other at the 3rd step of the figure in the corners of two squares. For smaller angles α, there are still no intersections at the 3rd step.



For $\alpha \approx 27.7°$, i.e. a ≈ 0.525 · b, two branches touch at the 4th step; overlaps occur at larger angles. For $\alpha \approx 24.8°$, i.e. a ≈ 0.462 · b, two branches touch at the 5th step; at larger angles overlaps occur.

There is still a lot to investigate!

Although the total area of the triangles and squares grows beyond all limits, the actual area occupied is limited, cf.

- https://en.wikipedia.org/wiki/Pythagoras_tree_(fractal)

✱ A 17.9:

The area of an equilateral triangle with side length s can be calculated by $A = \frac{1}{4} \cdot \sqrt{3} \cdot s^2$, because for the area we have A = ½ · s · h and the altitude h can be calculated with the Pythagorean theorem from

h² + (½ · s)² = s², thus h² = ¾ · s².

This also results from the following general consideration: The area A_n of a regular n-sided polygon is the n-fold of the area of the n isosceles triangles with base side s and leg r, where applies (cf. Chap. 1):

$$h = \frac{s}{2 \cdot \tan\left(\frac{180°}{n}\right)}, \text{ thus } A_n = n \cdot \frac{1}{2} \cdot s \cdot h = n \cdot \frac{s^2}{4 \cdot \tan\left(\frac{180°}{n}\right)}.$$

For n = 3 we have $\tan\left(\frac{180°}{n}\right) = \sqrt{3}$ and therefore $A_3 = 3 \cdot \frac{s^2}{4 \cdot \sqrt{3}} = \frac{s^2}{4} \cdot \sqrt{3}$.

For n = 5 we have $\tan\left(\frac{180°}{n}\right) = \sqrt{5 - 2\sqrt{5}}$ and therefore

$$A_5 = 5 \cdot \frac{s^2}{4 \cdot \sqrt{5 - 2\sqrt{5}}} = \frac{5}{4} \cdot \frac{\sqrt{5 + 2\sqrt{5}}}{\sqrt{5 - 2\sqrt{5}} \cdot \sqrt{5 + 2\sqrt{5}}} \cdot s^2 = \frac{5}{4} \cdot \frac{\sqrt{5 + 2\sqrt{5}}}{\sqrt{25 - 20}} \cdot s^2$$

$$= \frac{5}{4} \cdot \frac{\sqrt{5 + 2\sqrt{5}}}{\sqrt{5}} \cdot s^2 = \frac{1}{4} \cdot \sqrt{5 + 2\sqrt{5}} \cdot \sqrt{5} \cdot s^2 = \frac{1}{4} \cdot \sqrt{25 + 10\sqrt{5}} \cdot s^2$$

For n = 6 we have $\tan\left(\frac{180°}{n}\right) = \frac{\sqrt{3}}{3}$ and $A_6 = 6 \cdot \frac{3 \cdot s^2}{4 \cdot \sqrt{3}} = \frac{3}{2} \cdot \sqrt{3} \cdot s^2$.

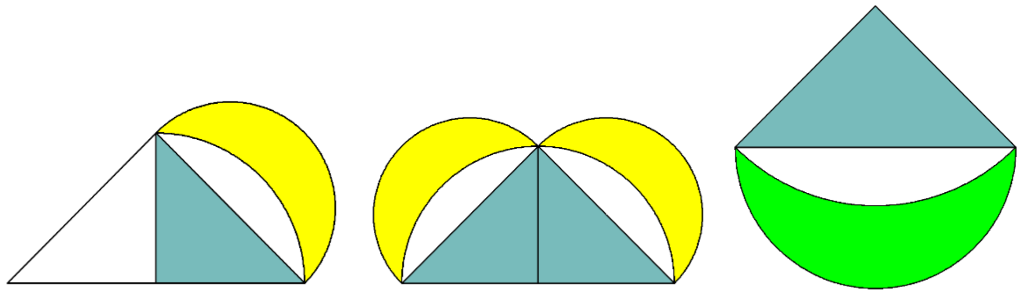
For n = 8 we have $\tan\left(\frac{180°}{n}\right) = \sqrt{2} - 1$ and $A_8 = 8 \cdot \frac{s^2}{4 \cdot (\sqrt{2} - 1)} = 2 \cdot \frac{s^2}{\sqrt{2} - 1} \cdot \frac{\sqrt{2} + 1}{\sqrt{2} + 1} = 2 \cdot (\sqrt{2} + 1) \cdot s^2$.

For semicircles we have: A = ½ · (½ · s)² · π = $\frac{\pi}{8} \cdot s^2$.

✱ A 17.10:

The figure on the left is a special case of an isosceles right triangle that has been doubled. The four lunes above the sides of a square are equal in area to the square.

In the illustration on the right, the generalisation of the Pythagorean theorem is applied to similar figures on the sides. Here we are dealing with little lunes above the hypotenuse of isosceles right-angled triangles. You can also formulate it like this: The two yellow-coloured lunes are equal in area to half of the triangle. The whole triangle is equal in area to the green-coloured lune.



✱ A 17.11:

The regular hexagon consists of six equilateral triangles whose sides are equal to the radius r of the circumcircle of the regular hexagon. The 6 semicircles above the sides of the hexagon have a total area of $A_{sc} = 6 \cdot \frac{1}{2} \cdot \frac{r^2}{4} \cdot \pi = \frac{3}{4} \cdot r^2 \cdot \pi$, the hexagon an area of $A_6 = 6 \cdot \frac{\sqrt{3}}{4} \cdot r^2$, together we have an area of $A_{total} = \frac{3}{4} \cdot r^2 \cdot \left(\pi + 2\sqrt{3}\right)$.

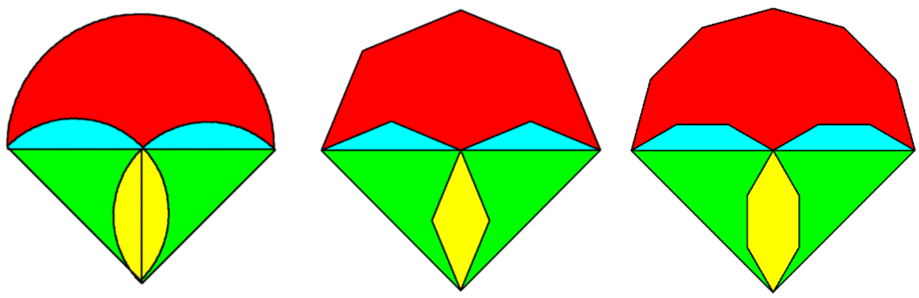
Subtracting from this the area of the circle with radius r gives the green-coloured lunes:

$$A = \frac{3}{4} \cdot r^2 \cdot \left(\pi + 2\sqrt{3}\right) - \pi \cdot r^2 = \frac{3}{4} \cdot r^2 \cdot \left(\pi + 2\sqrt{3} - \frac{4}{3}\pi\right) = \frac{3}{4} \cdot r^2 \cdot \left(2\sqrt{3} - \frac{1}{3}\pi\right)$$

This is the same as the area of the yellow-coloured

$$A_6 = 6 \cdot \frac{\sqrt{3}}{4} \cdot r^2 - \left(\frac{r}{2}\right)^2 \cdot \pi = \frac{3}{4} \cdot \left(2 \cdot \sqrt{3} - \frac{1}{3}\pi\right) \cdot r^2$$

✱ A 17.12:



We consider an isosceles right-angled triangle: The altitude of this triangle has the same length as half the hypotenuse and consequently the light blue coloured areas are the same size as the yellow coloured areas.

In contrast to lunes of Hippocrates, the semicircles on the legs are folded upwards. The blue areas then belong both to the area of the semicircles on the hypotenuse and to the folded-up semicircles on the sides and can therefore be omitted. However, since the two yellow areas belong to both semicircles, you need compensation areas – and these are precisely the blue areas, which are just as large as the yellow ones.

Therefore we have: red = light-blue + yellow + green

Instead of the semicircles, one can also accordingly consider half regular 8-sided polygons. And since the number of lunes on the sides of the half isosceles-rectangular triangles is even, it follows for the regular polygon that the number of vertices must be divisible by 4.

✱ A 17.13:

Figure on the left: Not drawn is the altitude to the hypotenuse and the two semicircles. The following applies to the areas of these two semicircles that are not drawn:

left: A_{sc} = light blue – violet right: A_{sc} = green – olive green

thus: light blue + olive green = green + violet.

Figure on the right: The two orange coloured semicircles on the hypotenuse segment p and the altitude h which is drawn downwards are together the same size as the yellow coloured semicircle, i.e.

yellow = orange.

Since the entire semicircle below the hypotenuse (i.e. olive green + orange) is as large as the two semicircles above the legs (yellow + green), it follows:

green = olive green.

✱ A 17.14:

The area of the light blue semicircle on the hypotenuse is as large as that of the two dark blue semicircles on he other sides. These blue semicircles in turn are the same size as the green and red semicircles in the right-angled triangle (divided by the altitude), i.e.

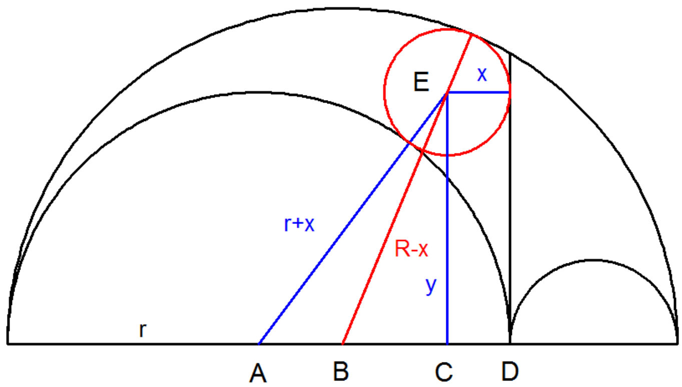
green + red = light blue, i.e. light blue - green = yellow.

If you cut the green semicircles out of the light blue semicircle, you have the yellow coloured area, i.e.

yellow = red.

✱ A 17.15:

The proof can be done, for example, with the help of the Pythagorean theorem.



If we denote the radius of the semicircle with R and the radius of the larger semicircle below it with r, the radius of the twin circle with x and the distance of the centre of the twin circle from the diameter with y = CE, then the following relations can be established for y^2:

ΔADE: $|AC|^2 + y^2 = |AE|^2$, with $|AC| = |AD| - x = r - x$ and $|AE| = r + x$,

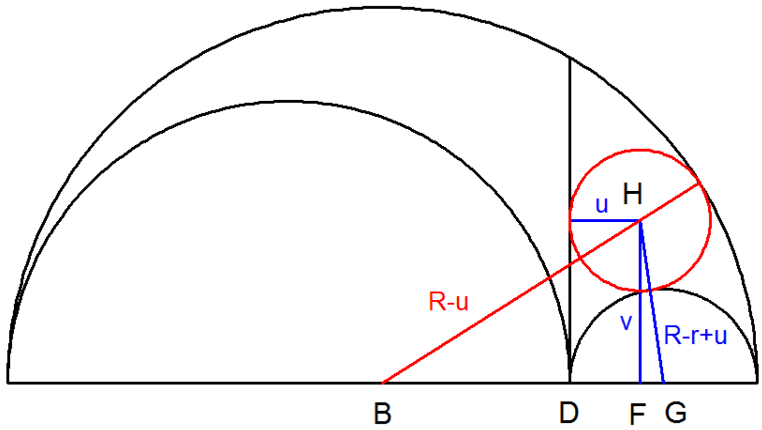
therefore: $y^2 = (r + x)^2 - (r - x)^2 = 4 \cdot rx$

ΔBCE: $|BC|^2 + y^2 = |BE|^2$, with $|BC| = |BD| - x = 2r - R - x$ and $|BE| = R - x$,

therefore: $y^2 = (R - x)^2 - (2r - R - x)^2 = 4 \cdot (Rr + rx - Rx - r^2)$.

From this we get: $rx = Rr + rx - Rx - r^2 \Leftrightarrow Rx = Rr - r^2$ and therefore

$$x = \frac{r \cdot (R - r)}{R}$$



In the part on the right side applies:

ΔBFH: $|BF|^2 + v^2 = |BH|^2$, with $|BF| = 2r - R + u$ and $|BH| = R - u$,

therefore: $v^2 = (R - u)^2 - (2r - R + u)^2 = 4 \cdot (rR + ru - r^2)$

ΔFGH: $|FG|^2 + v^2 = |GH|^2$, with $|FG| = R - r - u$ and $|GH| = R - r + u$,

therefore: $v^2 = (R - r + u)^2 - (R - r - u)^2 = 4 \cdot (R - r) \cdot u$

From this we get: $Ru - ru = Rr + ru - r^2 \Leftrightarrow Ru = Rr - r^2$ and therefore

$u = \dfrac{r \cdot (R - r)}{R}$. Thus we have $x = u$.

In the graphic visible on the Italian stamp, the proof is provided with the help of considerations about similarity.

✱ A 17.16:

The green coloured area is composed of the area of a semicircle with radius r, from which two semicircles with radius r/3 have been removed and a semicircle with radius r/3 has been added.

Therefore the area applies: ½ · π · $\frac{1}{2} \cdot \pi \cdot \left(r^2 - 2 \cdot \left(\frac{r}{3}\right)^2 + \left(\frac{r}{3}\right)^2 \right) = \frac{1}{2} \cdot \pi \cdot \frac{8}{9} \cdot r^2 = \pi \cdot \frac{4}{9} \cdot r^2 = \pi \cdot \left(\frac{2}{3} r\right)^2$

The light blue coloured circle has the diameter r + r/3, i.e. the radius $\frac{2}{3} r$. The area of this circle is therefore the same as that of the green coloured area in the figure on the left. If you colour the non-coloured area in the figure on the right green, then the total green coloured area matches that in the first figure, if you colour it light blue, then the total light blue coloured area matches that in the second figure.

✱ A 17.17:

An isosceles-rectangular triangle is considered, which is divided by an altitude into two congruent partial triangles, which are themselves isosceles-rectangular. This process can be repeated as often as desired. The line begins at the upper endpoint of the altitude and leads to the centre point of the opposite hypotenuse, which is again the upper endpoint of an altitude and so on. In the next step, the triangle lying to the left of the altitude is selected.

As was shown in A 17.10, a lune on the hypotenuse has the same area as the triangle.

The initial triangle has the area $A = ½ \cdot g \cdot h = ½ \cdot (2h) \cdot h = h^2 = 1$. Because of the continued bisection, the next triangles in the sequence each have half the area. As explained in chapter 8, the infinite sum

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots$ has the value 2.

Note: The line has the length

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{16}}+\frac{1}{\sqrt{32}}+\ldots=\frac{1}{1-\frac{1}{\sqrt{2}}}=\frac{1}{\frac{\sqrt{2}-1}{\sqrt{2}}}=\frac{\sqrt{2}}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}=2+\sqrt{2}$$

✱ A 17.18:

(1) The quadrilateral is actually a triangle that is divided into two right-angled triangles by the altitude h_c on c. Here applies: $h_c^2 = a^2 - p^2 = b^2 - q^2$, therefore $a^2 + q^2 = b^2 + p^2$.

(2) If x = y, the quadrilateral is axisymmetrical with regard to a diagonal, e.g. in the case of a symmetrical kite, then a = d and b = c apply. Then the condition $a^2 + c^2 = b^2 + d^2$ is fulfilled a fortiori.

✱ A 17.19:

The next sums of two square numbers for which more than one representation exist:

$125 = 5^2 + 10^2 = 2^2 + 11^2$ $\quad$ $130 = 3^2 + 11^2 = 7^2 + 9^2$

$145 = 1^2 + 12^2 = 8^2 + 9^2$ $\quad$ $170 = 1^2 + 13^2 = 7^2 + 11^2$

$185 = 4^2 + 13^2 = 8^2 + 11^2$ $\quad$ $200 = 2^2 + 14^2 = 10^2 + 10^2$

$205 = 6^2 + 13^2 = 3^2 + 14^2$ $\quad$ $221 = 5^2 + 14^2 = 10^2 + 11^2$

$250 = 5^2 + 15^2 = 9^2 + 13^2$ $\quad$ $260 = 2^2 + 16^2 = 8^2 + 14^2$

The fact that numbers with the final digit 0 or 5 predominate here can be explained as follows:

The only possible final digits of square numbers are:

$0^2 \equiv 0,\ 1^2 \equiv 1,\ 9^2 \equiv 1,\ 2^2 \equiv 4,\ 8^2 \equiv 4,\ 3^2 \equiv 9,\ 7^2 \equiv 9,\ 4^2 \equiv 6,\ 6^2 \equiv 6,\ 5^2 \equiv 5 \pmod{10}$

If one adds square numbers with these final digits, then the following combinations result:

	0	1	4	5	6	9
0	0	1	4	5	6	9
1	1	2	5	6	7	0
4	4	5	8	9	0	3
5	5	6	9	0	1	4
6	6	7	0	1	2	5
9	9	0	3	4	5	8

✱ A 17.20:

(1) In example 1 the triples (5 ; 12 ; 13) and the 3-fold of the primitive triple (3 ; 4 ; 5) were considered, in example 2 the triples (5 ; 12 ; 13) and the 4-fold of the primitive triple (3 ; 4 ; 5), in example 3, the 5-fold of the primitive triple (3 ; 4 ; 5) and the 3-fold of the primitive triple (5 ; 12 ; 13), in example 4, the 5-fold of the primitive triple (3 ; 4 ; 5) and the 4-fold of the primitive triple (5 ; 12 ; 13).

(2) If one considers the 8-fold of the primitive triple (3 ; 4 ; 5) and the 3-fold of the primitive triple (8 ; 15 ; 17), then the triples (24 ; 32 ; 40) and (24 ; 45 ; 51) result.

If we consider the 5-fold of the primitive triple (3 ; 4 ; 5) and the primitive triple (8 ; 15 ; 17) itself, then the triples (15 ; 20 ; 25) and (8 ; 15 ; 17) result. If one considers the 2-fold of the primitive triple (3 ; 4 ; 5) and the primitive triple (8 ; 15 ; 17) itself, then the triples (6 ; 8 ; 10) and (8 ; 15 ; 17) result. If one considers the 15-fold of the primitive triple (3 ; 4 ; 5) and the 4-fold of the primitive triple (8 ; 15 ; 17), then the triples (45 ; 60 ; 75) and (32 ; 60 ; 68) result.

(3) If one considers the 7-fold of the primitive triple (3 ; 4 ; 5) and the 3-fold of the primitive triple (7 ; 24 ; 25), then the triples (21 ; 28 ; 35) and (24 ; 72 ; 75) result.

If one considers the 8-fold of the primitive triple (3 ; 4 ; 5) and the primitive triple (7 ; 24 ; 25) itself, then the triples (24 ; 32 ; 40) and (7; 24 ; 25) result. If one considers the 7-fold of the primitive triple (3 ; 4 ; 5) and the 4-fold of the primitive triple (7 ; 24 ; 25), then the triples (21 ; 28 ; 35) and (28 ; 96 ; 100) result. If we consider the 6-fold of the primitive triple (3 ; 4 ; 5) and the primitive triple (7 ; 24 ; 25) itself, we obtain the triples (18 ; 24 ; 30) and (7 ; 24 ; 25).