JAMSHID AL-KASHI (1380 – June 22, 1429)

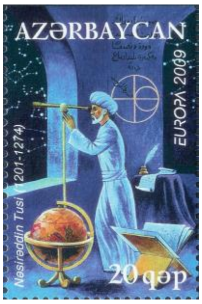
by HEINZ KLAUS STRICK, Germany

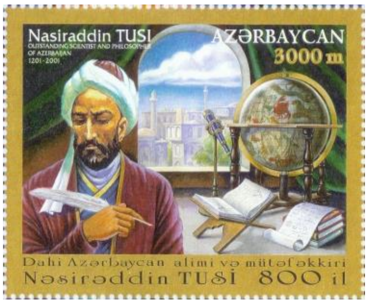
GHYATH-AL-DIN JAMSHID AL-KASHI is considered the last great mathematician of the Islamic Middle Ages.

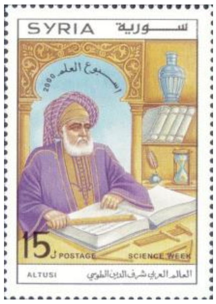
Born in Kashan (in present-day Iran), as a child he experienced the conquest of the region by Mongol armies under the regent TIMUR. It was only after TIMUR's death in 1405 and the division of the empire that cultural and scientific life blossomed again. The heir to TIMUR's throne, SHAK ROKH, appointed his son ULUGH BEG (1394 – 1449, the name means "Grand Prince") as the new ruler of the country's capital, the city of Samarkand (today in Uzbekistan).

The first reports about AL-KASHI date back to 1406. He worked in the surroundings of Kashan as an astronomer and mathematician and observed a lunar eclipse in 1406 and shortly afterwards he finished a work on the size of the cosmos, in which he critically examined the estimates of his predecessors.

In 1414 he published the corrected and expanded version of the great astronomical tables of NASIR AL-DIN AL-TUSI (1201 – 1274), which he dedicated – as was customary – to the ruler.







The *Khagani Zij* (Tables of the Khan) contained – in addition to extensive trigonometric tables – a star catalogue as well as information on the movements of the sun, moon and planets, described by equatorial coordinates (the coordinate system given by the equatorial plane and the polar axis) and also by ecliptic coordinates.

In 1416, a treatise on astronomical instruments followed, including AL-KASHI's inventions, e.g. a device for predicting planetary conjunctions.

Around 1420, ULUGH BEG invited the sixty best scientists of the country to come to Samarkand as researchers and teachers at the newly founded university (in Arabic *madrasa*). AL-KASHI was certainly the most capable of them. Later he would be called the second PTOLEMY.

From letters written by AL-KASHI to his father, which have been preserved, we know that AL-KASHI and his ruler ULUGH BEG treated each other with great mutual respect, for ULUGH BEG was also an outstanding astronomer.

In 1424, he began the construction of the great observatory in Samarkand, whose completion in 1437 AL-KASHI did not live to see.

The fact that astronomy experienced a new flowering in Islamic countries was related to the need for precise observations and geometrical calculations resulting from religious instructions, e.g. determining the direction of Mecca or the beginning of an Islamic month (the moment when the crescent moon is visible for the first time after a new moon).

In 1424 and 1427, two works by AL-KASHI appeared that are among the most important in the history of Arabic-Islamic mathematics: *Treatise on the circumference of a circle* and *The Key to Arithmetic*, a compendium of mathematics that was still used as a textbook in Persian schools until the 17th century.

In the *Treatise on the circumference of a circle*, AL-KASHI pursued the ambitious goal of determining π so precisely that the error occurring in a circle the size of the celestial sphere (estimate at the time: 600,000 earth diameters) should be smaller than the *width of a horse's hair* (an old unit of measurement, equivalent to 0.7 mm). He performed the calculations in the sexagesimal system on a regular polygon with $805,306,368 = 3 \cdot 2^{28}$ vertices, with 10-digit accuracy (corresponding to 16-digit accuracy in the decimal system), thus surpassing all calculations performed until then, e.g. those of ZU CHONGZHI from the 5th century.

The Key to Arithmetic was written as a textbook for students. The book was intended to provide future astronomers, surveyors, architects, accountants and merchants with the necessary knowledge of mathematics.

In the first part of the work, AL-KASHI dealt with arithmetic with natural numbers (using the number notation introduced from India); in the second part with arithmetic with fractions in the sexagesimal system, as was customary in the Arabic-Islamic cultural area at the time, but also with fractions in the decimal system, which was not rediscovered in Europe until 150 years later by SIMON STEVIN (1548 – 1620).

The fact that fractions were also written down in the base-60 system is connected with the calculation of angles and times (1° = 60'; 1' = 60'', 1 hr = 60 min); even the currency used at that time contained this 60s subdivision (1 *Dirham* = 60 *Fulus*).

The types of arithmetic covered in the book also included written root extraction. AL-KASHI demonstrated the extraction of the 5th root of 44,240,899,506,197 – using the binomial theorem and the binomial coefficients – and obtains $536 + \dfrac{21}{414{,}237{,}740{,}281}$, calculating the fraction by linear interpolation ($537^5 - 536^5 = 414{,}237{,}740{,}281$).

The other parts of the work dealt with the calculation of areas and volumes, including regular and semi-regular polyhedra (Platonic and Archimedean solids) as well as other forms used in Islamic architecture (domes), with the problem of the irrationality of numbers, with the summation of finite sequences of numbers as well as with problems from number theory.

In France, a theorem is still called *Le théorème d'AL-KASHI* (in our country known as the *cosine rule*), because AL-KASHI was the first to explain how the theorem, which was already known to other Arabic-Islamic mathematicians, could be applied in triangulation (measurement of triangles).

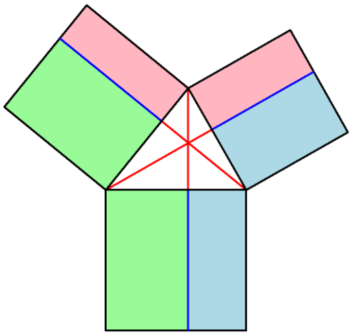
$a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha)$

$b^2 = c^2 + a^2 - 2ca \cdot \cos(\beta)$

$c^2 = a^2 + b^2 - 2ab \cdot \cos(\gamma)$

Proof of the theorem, e. g.

$a^2 + b^2 = (green + pink) + (pink + blue)$
$= (green + blue) + 2 \cdot pink = c^2 + 2ab \cdot \cos(\gamma)$



AL-KASHI's last masterpiece was *The Treatise on Chord and Sine* from 1427, in which he succeeded in calculating a very precise value for sin(1°), from which ULUGH BEG later derived approximate values for other angles:
From the knowledge of $\sin(30°) = \frac{1}{2}$, $\cos(30°) = \frac{\sqrt{3}}{2}$ (from the equilateral triangle) and

$\cos(36°) = \frac{\sqrt{5}+1}{4}$, $\sin(36°) = \frac{\sqrt{10-2\sqrt{5}}}{4}$ (from the regular 10-sided figure),

he determined exact values of $\sin(6°) = \sin(36° - 30°)$ and $\cos(6°) = \cos(36° - 30°)$
with the help of the addition theorems, then from this with the help of

$\sin(3°) = \sqrt{\frac{1-\cos(6°)}{2}}$ and $\cos(3°) = \sqrt{\frac{1+\cos(6°)}{2}}$ approximate values for sin(3°) and cos(3°).

Finally, in order to be able to calculate $\sin(1°)$ and $\cos(1°)$, he used the *triple angle formulae* which he had discovered:

$\sin(3\alpha) = 3 \cdot \sin(\alpha) - 4 \cdot \sin^3(\alpha)$ and $\cos(3\alpha) = 4 \cdot \cos^3(\alpha) - 3 \cdot \cos(\alpha)$.

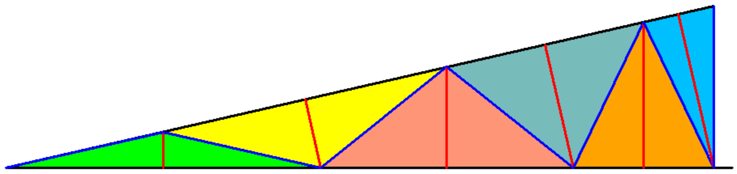
AL-KASHI interpreted the formulae to mean that for the *trisection of the angle* of 3° a cubic equation had to be solved: $x^3 + b = a \cdot x$, where $x = \sin(1°)$; $a = \frac{3}{4}$; $b = \frac{1}{4} \cdot \sin(3°)$.

He determined this solution iteratively, using $x = \dfrac{b + x^3}{a} \approx \dfrac{b}{a}$ as a first approximate solution, which one could put back into the equation: $x_2 = \dfrac{b + x_1^3}{a}$, then $x_3 = \dfrac{b + x_2^3}{a}$ etc.

Finally, AL-KASHI obtained a result accurate to 18 decimal places:

sin(1°) = 0.017452406437283571.

The triple angle formulae mentioned can be derived from the figure shown below:



The segments of the zigzag line each have length 1. In the figure, from left to right, the angles $\alpha, 2\alpha, 3\alpha$ occur; the following ratio equations can be read from similar partial figures:

$$\frac{\cos(\alpha)}{1} = \frac{1+\cos(2\alpha)}{2\cos(\alpha)} = \frac{2\cos(\alpha)+\cos(3\alpha)}{1+2\cos(2\alpha)} \quad \text{and} \quad \frac{\sin(\alpha)}{1} = \frac{\sin(2\alpha)}{2\cos(\alpha)} = \frac{\sin(3\alpha)}{1+2\cos(2\alpha)}$$

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